CHAPTER

6

The Laplace Transform

Many practical engineering problems involve mechanical or electrical systems acted on by discontinuous or impulsive forcing terms. For such problems the methods described in Chapter 3 are often rather awkward to use. Another method that is especially well suited to these problems, although useful much more generally, is based on the Laplace transform. In this chapter we describe how this important method works, emphasizing problems typical of those arising in engineering applications.

6.1 Definition of the Laplace Transform

Among the tools that are very useful for solving linear differential equations are **integral transforms**. An integral transform is a relation of the form

$$F(s) = \int_{\alpha}^{\beta} K(s,t) f(t)\, dt, \tag{1}$$

where $K(s,t)$ is a given function, called the **kernel** of the transformation, and the limits of integration α and β are also given. It is possible that $\alpha = -\infty$ or $\beta = \infty$, or both. The relation (1) transforms the function f into another function F, which is called the transform of f. The general idea in using an integral transform to solve a differential equation is as follows: Use the relation (1) to transform a problem for an unknown function f into a simpler problem for F, then solve this simpler problem to find F, and finally recover the desired function f from its transform F. This last step is known as "inverting the transform."

There are several integral transforms that are useful in applied mathematics, but in this chapter we consider only the Laplace[1] transform. This transform is defined in the following way. Let $f(t)$ be given for $t \geq 0$, and suppose that f satisfies certain conditions to be stated a little later. Then the Laplace transform of f, which we will denote by $\mathcal{L}\{f(t)\}$ or by $F(s)$, is defined by the equation

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt. \tag{2}$$

The Laplace transform makes use of the kernel $K(s, t) = e^{-st}$. Since the solutions of linear differential equations with constant coefficients are based on the exponential function, the Laplace transform is particularly useful for such equations.

Since the Laplace transform is defined by an integral over the range from zero to infinity, it is useful to review some basic facts about such integrals. In the first place, an integral over an unbounded interval is called an **improper integral**, and is defined as a limit of integrals over finite intervals; thus

$$\int_a^\infty f(t)\, dt = \lim_{A\to\infty} \int_a^A f(t)\, dt, \tag{3}$$

where A is a positive real number. If the integral from a to A exists for each $A > a$, and if the limit as $A \to \infty$ exists, then the improper integral is said to **converge** to that limiting value. Otherwise the integral is said to **diverge**, or to fail to exist. The following examples illustrate both possibilities.

EXAMPLE 1

Let $f(t) = e^{ct}$, $t \geq 0$, where c is a real nonzero constant. Then

$$\begin{aligned}\int_0^\infty e^{ct}\, dt &= \lim_{A\to\infty} \int_0^A e^{ct}\, dt = \lim_{A\to\infty} \left.\frac{e^{ct}}{c}\right|_0^A \\ &= \lim_{A\to\infty} \frac{1}{c}(e^{cA} - 1).\end{aligned}$$

It follows that the improper integral converges if $c < 0$, and diverges if $c > 0$. If $c = 0$, the integrand $f(t)$ is the constant function with value 1, and the integral again diverges.

EXAMPLE 2

Let $f(t) = 1/t$, $t \geq 1$. Then

$$\int_1^\infty \frac{dt}{t} = \lim_{A\to\infty} \int_1^A \frac{dt}{t} = \lim_{A\to\infty} \ln A.$$

Since $\lim_{A\to\infty} \ln A = \infty$, the improper integral diverges.

[1] The Laplace transform is named for the eminent French mathematician P. S. Laplace, who studied the relation (2) in 1782. However, the techniques described in this chapter were not developed until a century or more later. They are due mainly to Oliver Heaviside (1850–1925), an innovative but unconventional English electrical engineer, who made significant contributions to the development and application of electromagnetic theory.

EXAMPLE 3

Let $f(t) = t^{-p}$, $t \geq 1$, where p is a real constant and $p \neq 1$; the case $p = 1$ was considered in Example 2. Then

$$\int_1^\infty t^{-p}\,dt = \lim_{A\to\infty}\int_1^A t^{-p}\,dt = \lim_{A\to\infty}\frac{1}{1-p}(A^{1-p} - 1).$$

As $A \to \infty$, $A^{1-p} \to 0$ if $p > 1$, but $A^{1-p} \to \infty$ if $p < 1$. Hence $\int_1^\infty t^{-p}\,dt$ converges for $p > 1$, but (incorporating the result of Example 2) diverges for $p \leq 1$. These results are analogous to those for the infinite series $\sum_{n=1}^{\infty} n^{-p}$.

Before discussing the possible existence of $\int_a^\infty f(t)\,dt$, it is helpful to define certain terms. A function f is said to be **piecewise continuous** on an interval $\alpha \leq t \leq \beta$ if the interval can be partitioned by a finite number of points $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ so that

1. f is continuous on each open subinterval $t_{i-1} < t < t_i$.
2. f approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval.

In other words, f is piecewise continuous on $\alpha \leq t \leq \beta$ if it is continuous there except for a finite number of jump discontinuities. If f is piecewise continuous on $\alpha \leq t \leq \beta$ for every $\beta > \alpha$, then f is said to be piecewise continuous on $t \geq \alpha$. An example of a piecewise continuous function is shown in Figure 6.1.1.

If f is piecewise continuous on the interval $a \leq t \leq A$, then it can be shown that $\int_a^A f(t)\,dt$ exists. Hence, if f is piecewise continuous for $t \geq a$, then $\int_a^A f(t)\,dt$ exists for each $A > a$. However, piecewise continuity is not enough to ensure convergence of the improper integral $\int_a^\infty f(t)\,dt$, as the preceding examples show.

If f cannot be integrated easily in terms of elementary functions, the definition of convergence of $\int_a^\infty f(t)\,dt$ may be difficult to apply. Frequently, the most convenient way to test the convergence or divergence of an improper integral is by the following comparison theorem, which is analogous to a similar theorem for infinite series.

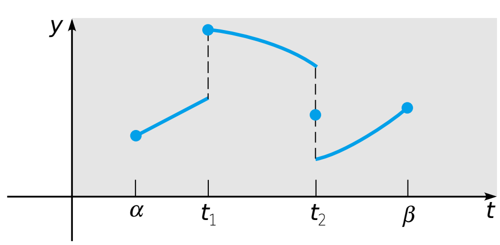

FIGURE 6.1.1 A piecewise continuous function.

Theorem 6.1.1 If f is piecewise continuous for $t \geq a$, if $|f(t)| \leq g(t)$ when $t \geq M$ for some positive constant M, and if $\int_M^\infty g(t)\,dt$ converges, then $\int_a^\infty f(t)\,dt$ also converges. On the other hand, if $f(t) \geq g(t) \geq 0$ for $t \geq M$, and if $\int_M^\infty g(t)\,dt$ diverges, then $\int_a^\infty f(t)\,dt$ also diverges.

The proof of this result from the calculus will not be given here. It is made plausible, however, by comparing the areas represented by $\int_M^\infty g(t)\,dt$ and $\int_M^\infty |f(t)|\,dt$. The functions most useful for comparison purposes are e^{ct} and t^{-p}, which were considered in Examples 1, 2, and 3.

We now return to a consideration of the Laplace transform $\mathcal{L}\{f(t)\}$ or $F(s)$, which is defined by Eq. (2) whenever this improper integral converges. In general, the parameter s may be complex, but for our discussion we need consider only real values of s. The foregoing discussion of integrals indicates that the Laplace transform F of a function f exists if f satisfies certain conditions, such as those stated in the following theorem.

Theorem 6.1.2 Suppose that

1. f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive A.
2. $|f(t)| \leq Ke^{at}$ when $t \geq M$. In this inequality K, a, and M are real constants, K and M necessarily positive.

Then the Laplace transform $\mathcal{L}\{f(t)\} = F(s)$, defined by Eq. (2), exists for $s > a$.

To establish this theorem it is necessary to show only that the integral in Eq. (2) converges for $s > a$. Splitting the improper integral into two parts, we have

$$\int_0^\infty e^{-st} f(t)\,dt = \int_0^M e^{-st} f(t)\,dt + \int_M^\infty e^{-st} f(t)\,dt. \tag{4}$$

The first integral on the right side of Eq. (4) exists by hypothesis (1) of the theorem; hence the existence of $F(s)$ depends on the convergence of the second integral. By hypothesis (2) we have, for $t \geq M$,

$$|e^{-st} f(t)| \leq Ke^{-st}e^{at} = Ke^{(a-s)t},$$

and thus, by Theorem 6.1.1, $F(s)$ exists provided that $\int_M^\infty e^{(a-s)t}\,dt$ converges. Referring to Example 1 with c replaced by $a - s$, we see that this latter integral converges when $a - s < 0$, which establishes Theorem 6.1.2.

Unless the contrary is specifically stated, in this chapter we deal only with functions satisfying the conditions of Theorem 6.1.2. Such functions are described as piecewise continuous, and of **exponential order** as $t \to \infty$. The Laplace transforms of some important elementary functions are given in the following examples.

EXAMPLE 4

Let $f(t) = 1$, $t \geq 0$. Then

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \frac{1}{s}, \qquad s > 0.$$

EXAMPLE 5

Let $f(t) = e^{at}$, $t \geq 0$. Then

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt$$
$$= \frac{1}{s-a}, \qquad s > a.$$

EXAMPLE 6

Let $f(t) = \sin at$, $t \geq 0$. Then

$$\mathcal{L}\{\sin at\} = F(s) = \int_0^\infty e^{-st} \sin at\,dt, \qquad s > 0.$$

Since

$$F(s) = \lim_{A\to\infty} \int_0^A e^{-st} \sin at\,dt,$$

upon integrating by parts we obtain

$$F(s) = \lim_{A\to\infty} \left[-\frac{e^{-st}\cos at}{a}\bigg|_0^A - \frac{s}{a}\int_0^A e^{-st}\cos at\,dt \right]$$
$$= \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-st}\cos at\,dt.$$

A second integration by parts then yields

$$F(s) = \frac{1}{a} - \frac{s^2}{a^2}\int_0^\infty e^{-st}\sin at\,dt$$
$$= \frac{1}{a} - \frac{s^2}{a^2}F(s).$$

Hence, solving for $F(s)$, we have

$$F(s) = \frac{a}{s^2 + a^2}, \qquad s > 0.$$

Now let us suppose that f_1 and f_2 are two functions whose Laplace transforms exist for $s > a_1$ and $s > a_2$, respectively. Then, for s greater than the maximum of a_1 and a_2,

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = \int_0^\infty e^{-st}[c_1 f_1(t) + c_2 f_2(t)]\,dt$$
$$= c_1 \int_0^\infty e^{-st} f_1(t)\,dt + c_2 \int_0^\infty e^{-st} f_2(t)\,dt;$$

hence

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}. \tag{5}$$

Equation (5) is a statement of the fact that the Laplace transform is a *linear operator*. This property is of paramount importance, and we make frequent use of it later.

PROBLEMS

In each of Problems 1 through 4 sketch the graph of the given function. In each case determine whether f is continuous, piecewise continuous, or neither on the interval $0 \le t \le 3$.

1. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 2+t, & 1 < t \le 2 \\ 6-t, & 2 < t \le 3 \end{cases}$

2. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ (t-1)^{-1}, & 1 < t \le 2 \\ 1, & 2 < t \le 3 \end{cases}$

3. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 1, & 1 < t \le 2 \\ 3-t, & 2 < t \le 3 \end{cases}$

4. $f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 3-t, & 1 < t \le 2 \\ 1, & 2 < t \le 3 \end{cases}$

5. Find the Laplace transform of each of the following functions:
   (a) t
   (b) t^2
   (c) t^n, where n is a positive integer
6. Find the Laplace transform of $f(t) = \cos at$, where a is a real constant.

Recall that $\cosh bt = (e^{bt} + e^{-bt})/2$ and $\sinh bt = (e^{bt} - e^{-bt})/2$. In each of Problems 7 through 10 find the Laplace transform of the given function; a and b are real constants.

7. $\cosh bt$
8. $\sinh bt$
9. $e^{at}\cosh bt$
10. $e^{at}\sinh bt$

In each of Problems 11 through 14 recall that $\cos bt = (e^{ibt} + e^{-ibt})/2$ and $\sin bt = (e^{ibt} - e^{-ibt})/2i$. Assuming that the necessary elementary integration formulas extend to this case, find the Laplace transform of the given function; a and b are real constants.

11. $\sin bt$
12. $\cos bt$
13. $e^{at}\sin bt$
14. $e^{at}\cos bt$

In each of Problems 15 through 20, using integration by parts, find the Laplace transform of the given function; n is a positive integer and a is a real constant.

15. te^{at}
16. $t\sin at$
17. $t\cosh at$
18. $t^n e^{at}$
19. $t^2\sin at$
20. $t^2\sinh at$

In each of Problems 21 through 24 determine whether the given integral converges or diverges.

21. $\displaystyle\int_0^\infty (t^2+1)^{-1}\,dt$
22. $\displaystyle\int_0^\infty te^{-t}\,dt$
23. $\displaystyle\int_1^\infty t^{-2}e^t\,dt$
24. $\displaystyle\int_0^\infty e^{-t}\cos t\,dt$
25. Suppose that f and f' are continuous for $t \ge 0$, and of exponential order as $t \to \infty$. Show by integration by parts that if $F(s) = \mathcal{L}\{f(t)\}$, then $\lim_{s\to\infty} F(s) = 0$. The result is actually true under less restrictive conditions, such as those of Theorem 6.1.2.
26. **The Gamma Function.** The gamma function is denoted by $\Gamma(p)$ and is defined by the integral

$$\Gamma(p+1) = \int_0^\infty e^{-x}x^p\,dx. \tag{i}$$

The integral converges as $x \to \infty$ for all p. For $p < 0$ it is also improper because the integrand becomes unbounded as $x \to 0$. However, the integral can be shown to converge at $x = 0$ for $p > -1$.

(a) Show that for $p > 0$

$$\Gamma(p+1) = p\Gamma(p).$$

(b) Show that $\Gamma(1) = 1$.

(c) If p is a positive integer n, show that

$$\Gamma(n+1) = n!.$$

Since $\Gamma(p)$ is also defined when p is not an integer, this function provides an extension of the factorial function to nonintegral values of the independent variable. Note that it is also consistent to define $0! = 1$.

(d) Show that for $p > 0$

$$p(p+1)(p+2)\cdots(p+n-1) = \Gamma(p+n)/\Gamma(p).$$

Thus $\Gamma(p)$ can be determined for all positive values of p if $\Gamma(p)$ is known in a single interval of unit length, say, $0 < p \le 1$. It is possible to show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Find $\Gamma(\frac{3}{2})$ and $\Gamma(\frac{11}{2})$.

27. Consider the Laplace transform of t^p, where $p > -1$.

(a) Referring to Problem 26, show that

$$\mathcal{L}\{t^p\} = \int_0^\infty e^{-st} t^p \, dt = \frac{1}{s^{p+1}} \int_0^\infty e^{-x} x^p \, dx$$
$$= \Gamma(p+1)/s^{p+1}, \qquad s > 0.$$

(b) Let p be a positive integer n in (a); show that

$$\mathcal{L}\{t^n\} = n!/s^{n+1}, \qquad s > 0.$$

(c) Show that

$$\mathcal{L}\{t^{-1/2}\} = \frac{2}{\sqrt{s}} \int_0^\infty e^{-x^2} \, dx, \qquad s > 0.$$

It is possible to show that

$$\int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2};$$

hence

$$\mathcal{L}\{t^{-1/2}\} = \sqrt{\pi/s}, \qquad s > 0.$$

(d) Show that

$$\mathcal{L}\{t^{1/2}\} = \sqrt{\pi}/2s^{3/2}, \qquad s > 0.$$

6.2 Solution of Initial Value Problems

In this section we show how the Laplace transform can be used to solve initial value problems for linear differential equations with constant coefficients. The usefulness of the Laplace transform in this connection rests primarily on the fact that the transform of f' is related in a simple way to the transform of f. The relationship is expressed in the following theorem.

Theorem 6.2.1 Suppose that f is continuous and f' is piecewise continuous on any interval $0 \le t \le A$. Suppose further that there exist constants K, a, and M such that $|f(t)| \le Ke^{at}$ for $t \ge M$. Then $\mathcal{L}\{f'(t)\}$ exists for $s > a$, and moreover

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0). \tag{1}$$

To prove this theorem we consider the integral

$$\int_0^A e^{-st} f'(t)\,dt.$$

If f' has points of discontinuity in the interval $0 \le t \le A$, let them be denoted by $t_1, t_2, \dots, t_n$. Then we can write this integral as

$$\int_0^A e^{-st} f'(t)\,dt = \int_0^{t_1} e^{-st} f'(t)\,dt + \int_{t_1}^{t_2} e^{-st} f'(t)\,dt + \cdots + \int_{t_n}^{A} e^{-st} f'(t)\,dt.$$

Integrating each term on the right by parts yields

$$\int_0^A e^{-st} f'(t)\,dt = e^{-st} f(t)\Big|_0^{t_1} + e^{-st} f(t)\Big|_{t_1}^{t_2} + \cdots + e^{-st} f(t)\Big|_{t_n}^{A}$$
$$+ s\left[\int_0^{t_1} e^{-st} f(t)\,dt + \int_{t_1}^{t_2} e^{-st} f(t)\,dt + \cdots + \int_{t_n}^{A} e^{-st} f(t)\,dt\right].$$

Since f is continuous, the contributions of the integrated terms at $t_1, t_2, \dots, t_n$ cancel. Combining the integrals gives

$$\int_0^A e^{-st} f'(t)\,dt = e^{-sA} f(A) - f(0) + s\int_0^A e^{-st} f(t)\,dt.$$

As $A \to \infty$, $e^{-sA} f(A) \to 0$ whenever $s > a$. Hence, for $s > a$,

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0),$$

which establishes the theorem.

If f' and f'' satisfy the same conditions that are imposed on f and f', respectively, in Theorem 6.2.1, then it follows that the Laplace transform of f'' also exists for $s > a$ and is given by

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0). \tag{2}$$

Indeed, provided the function f and its derivatives satisfy suitable conditions, an expression for the transform of the nth derivative $f^{(n)}$ can be derived by successive applications of this theorem. The result is given in the following corollary.

Corollary 6.2.2 Suppose that the functions $f, f', \dots, f^{(n-1)}$ are continuous, and that $f^{(n)}$ is piecewise continuous on any interval $0 \le t \le A$. Suppose further that there exist constants K, a, and M such that $|f(t)| \le Ke^{at}$, $|f'(t)| \le Ke^{at}$, $\ldots$, $|f^{(n-1)}(t)| \le Ke^{at}$ for $t \ge M$. Then $\mathcal{L}\{f^{(n)}(t)\}$ exists for $s > a$ and is given by

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0). \tag{3}$$

We now show how the Laplace transform can be used to solve initial value problems. It is most useful for problems involving nonhomogeneous differential equations, as we will demonstrate in later sections of this chapter. However, we begin by looking at some homogeneous equations, which are a bit simpler. For example, consider the differential equation

$$y'' - y' - 2y = 0 \tag{4}$$

and the initial conditions

$$y(0) = 1, \qquad y'(0) = 0. \tag{5}$$

This problem is easily solved by the methods of Section 3.1. The characteristic equation is

$$r^2 - r - 2 = (r - 2)(r + 1) = 0, \tag{6}$$

and consequently the general solution of Eq. (4) is

$$y = c_1e^{-t} + c_2e^{2t}. \tag{7}$$

To satisfy the initial conditions (5) we must have $c_1 + c_2 = 1$ and $-c_1 + 2c_2 = 0$; hence $c_1 = \frac{2}{3}$ and $c_2 = \frac{1}{3}$, so that the solution of the initial value problem (4) and (5) is

$$y = \phi(t) = \tfrac{2}{3}e^{-t} + \tfrac{1}{3}e^{2t}. \tag{8}$$

Now let us solve the same problem by using the Laplace transform. To do this we must assume that the problem has a solution $y = \phi(t)$, which with its first two derivatives satisfies the conditions of Corollary 6.2.2. Then, taking the Laplace transform of the differential equation (4), we obtain

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0, \tag{9}$$

where we have used the linearity of the transform to write the transform of a sum as the sum of the separate transforms. Upon using the corollary to express $\mathcal{L}\{y''\}$ and $\mathcal{L}\{y'\}$ in terms of $\mathcal{L}\{y\}$, we find that Eq. (9) becomes

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - [s\mathcal{L}\{y\} - y(0)] - 2\mathcal{L}\{y\} = 0,$$

or

$$(s^2 - s - 2)Y(s) + (1 - s)y(0) - y'(0) = 0, \tag{10}$$

where $Y(s) = \mathcal{L}\{y\}$. Substituting for $y(0)$ and $y'(0)$ in Eq. (10) from the initial conditions (5), and then solving for $Y(s)$, we obtain

$$Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}. \tag{11}$$

We have thus obtained an expression for the Laplace transform $Y(s)$ of the solution $y = \phi(t)$ of the given initial value problem. To determine the function ϕ we must find the function whose Laplace transform is $Y(s)$, as given by Eq. (11).

This can be done most easily by expanding the right side of Eq. (11) in partial fractions. Thus we write

$$Y(s) = \frac{s-1}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1)+b(s-2)}{(s-2)(s+1)}, \tag{12}$$

where the coefficients a and b are to be determined. By equating numerators of the second and fourth members of Eq. (12), we obtain

$$s-1 = a(s+1) + b(s-2),$$

an equation that must hold for all s. In particular, if we set $s = 2$, then it follows that $a = \frac{1}{3}$. Similarly, if we set $s = -1$, then we find that $b = \frac{2}{3}$. By substituting these values for a and b, respectively, we have

$$Y(s) = \frac{1/3}{s-2} + \frac{2/3}{s+1}. \tag{13}$$

Finally, if we use the result of Example 5 of Section 6.1, it follows that $\frac{1}{3}e^{2t}$ has the transform $\frac{1}{3}(s-2)^{-1}$; similarly, $\frac{2}{3}e^{-t}$ has the transform $\frac{2}{3}(s+1)^{-1}$. Hence, by the linearity of the Laplace transform,

$$y = \phi(t) = \tfrac{1}{3}e^{2t} + \tfrac{2}{3}e^{-t}$$

has the transform (13) and is therefore the solution of the initial value problem (4), (5). Of course, this is the same solution that we obtained earlier.

The same procedure can be applied to the general second order linear equation with constant coefficients,

$$ay'' + by' + cy = f(t). \tag{14}$$

Assuming that the solution $y = \phi(t)$ satisfies the conditions of Corollary 6.2.2 for $n = 2$, we can take the transform of Eq. (14) and thereby obtain

$$a[s^2Y(s) - sy(0) - y'(0)] + b[sY(s) - y(0)] + cY(s) = F(s), \tag{15}$$

where $F(s)$ is the transform of $f(t)$. By solving Eq. (15) for $Y(s)$ we find that

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2+bs+c} + \frac{F(s)}{as^2+bs+c}. \tag{16}$$

The problem is then solved, provided that we can find the function $y = \phi(t)$ whose transform is $Y(s)$.

Even at this early stage of our discussion we can point out some of the essential features of the transform method. In the first place, the transform $Y(s)$ of the unknown function $y = \phi(t)$ is found by solving an *algebraic equation* rather than a *differential equation*, Eq. (10) rather than Eq. (4), or in general Eq. (15) rather than Eq. (14). This is the key to the usefulness of Laplace transforms for solving linear, constant coefficient, ordinary differential equations—the problem is reduced from a differential equation to an algebraic one. Next, the solution satisfying given initial conditions is automatically found, so that the task of determining appropriate values for the arbitrary constants in the general solution does not arise. Further, as indicated in Eq. (15), nonhomogeneous equations are handled in exactly the same way as homogeneous ones; it is not necessary to solve the corresponding homogeneous equation first. Finally, the method can be applied in the same way to higher order equations, as long as we assume that the solution satisfies the conditions of the corollary for the appropriate value of n.

Observe that the polynomial $as^2 + bs + c$ in the denominator on the right side of Eq. (16) is precisely the characteristic polynomial associated with Eq. (14). Since the use of a partial fraction expansion of $Y(s)$ to determine $\phi(t)$ requires us to factor this polynomial, the use of Laplace transforms does not avoid the necessity of finding roots of the characteristic equation. For equations of higher than second order this may be a difficult algebraic problem, particularly if the roots are irrational or complex.

The main difficulty that occurs in solving initial value problems by the transform technique lies in the problem of determining the function $y = \phi(t)$ corresponding to the transform $Y(s)$. This problem is known as the inversion problem for the Laplace transform; $\phi(t)$ is called the inverse transform corresponding to $Y(s)$, and the process of finding $\phi(t)$ from $Y(s)$ is known as inverting the transform. We also use the notation $\mathcal{L}^{-1}\{Y(s)\}$ to denote the inverse transform of $Y(s)$. There is a general formula for the inverse Laplace transform, but its use requires a knowledge of the theory of functions of a complex variable, and we do not consider it in this book. However, it is still possible to develop many important properties of the Laplace transform, and to solve many interesting problems, without the use of complex variables.

In solving the initial value problem (4), (5) we did not consider the question of whether there may be functions other than the one given by Eq. (8) that also have the transform (13). In fact, it can be shown that if f is a continuous function with the Laplace transform F, then there is no other continuous function having the same transform. In other words, there is essentially a one-to-one correspondence between functions and their Laplace transforms. This fact suggests the compilation of a table, such as Table 6.2.1, giving the transforms of functions frequently encountered, and vice versa. The entries in the second column of Table 6.2.1 are the transforms of those in the first column. Perhaps more important, the functions in the first column are the inverse transforms of those in the second column. Thus, for example, if the transform of the solution of a differential equation is known, the solution itself can often be found merely by looking it up in the table. Some of the entries in Table 6.2.1 have been used as examples, or appear as problems in Section 6.1, while others will be developed later in the chapter. The third column of the table indicates where the derivation of the given transforms may be found. While Table 6.2.1 is sufficient for the examples and problems in this book, much larger tables are also available (see the list of references at the end of the chapter). Transforms and inverse transforms can also be readily obtained electronically by using a computer algebra system.

Frequently, a Laplace transform $F(s)$ is expressible as a sum of several terms,

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s). \tag{17}$$

Suppose that $f_1(t) = \mathcal{L}^{-1}\{F_1(s)\}, \ldots, f_n(t) = \mathcal{L}^{-1}\{F_n(s)\}$. Then the function

$$f(t) = f_1(t) + \cdots + f_n(t)$$

has the Laplace transform $F(s)$. By the uniqueness property stated previously there is no other continuous function f having the same transform. Thus

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F_1(s)\} + \cdots + \mathcal{L}^{-1}\{F_n(s)\}; \tag{18}$$

that is, the inverse Laplace transform is also a linear operator.

In many problems it is convenient to make use of this property by decomposing a given transform into a sum of functions whose inverse transforms are already known or can be found in the table. Partial fraction expansions are particularly useful in this

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\dfrac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. t^n; $\quad n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
4. t^p, $\; p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
5. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 6
6. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 6.1; Prob. 8
8. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 6.1; Prob. 7
9. $e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}$, $\; n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 19
16. $\displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 28

connection, and a general result covering many cases is given in Problem 38. Other useful properties of Laplace transforms are derived later in this chapter.

As further illustrations of the technique of solving initial value problems by means of the Laplace transform and partial fraction expansions, consider the following examples.

EXAMPLE 1

Find the solution of the differential equation

$$y'' + y = \sin 2t, \tag{19}$$

satisfying the initial conditions

$$y(0) = 2, \qquad y'(0) = 1. \tag{20}$$

We assume that this initial value problem has a solution $y = \phi(t)$, which with its first two derivatives satisfies the conditions of Corollary 6.2.2. Then, taking the Laplace transform of the differential equation, we have

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 2/(s^2 + 4),$$

where the transform of $\sin 2t$ has been obtained from line 5 of Table 6.2.1. Substituting for $y(0)$ and $y'(0)$ from the initial conditions and solving for $Y(s)$, we obtain

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2 + 1)(s^2 + 4)}. \tag{21}$$

Using partial fractions we can write $Y(s)$ in the form

$$Y(s) = \frac{as + b}{s^2 + 1} + \frac{cs + d}{s^2 + 4} = \frac{(as + b)(s^2 + 4) + (cs + d)(s^2 + 1)}{(s^2 + 1)(s^2 + 4)}. \tag{22}$$

By expanding the numerator on the right side of Eq. (22) and equating it to the numerator in Eq. (21) we find that

$$2s^3 + s^2 + 8s + 6 = (a + c)s^3 + (b + d)s^2 + (4a + c)s + (4b + d)$$

for all s. Then, comparing coefficients of like powers of s, we have

$$a + c = 2, \qquad b + d = 1,$$
$$4a + c = 8, \qquad 4b + d = 6.$$

Consequently, $a = 2$, $c = 0$, $b = \frac{5}{3}$, and $d = -\frac{2}{3}$, from which it follows that

$$Y(s) = \frac{2s}{s^2 + 1} + \frac{5/3}{s^2 + 1} - \frac{2/3}{s^2 + 4}. \tag{23}$$

From lines 5 and 6 of Table 6.2.1, the solution of the given initial value problem is

$$y = \phi(t) = 2\cos t + \tfrac{5}{3}\sin t - \tfrac{1}{3}\sin 2t. \tag{24}$$

EXAMPLE 2

Find the solution of the initial value problem

$$y^{\text{iv}} - y = 0, \tag{25}$$

$$y(0) = 0, \qquad y'(0) = 1, \qquad y''(0) = 0, \qquad y'''(0) = 0. \tag{26}$$

In this problem we need to assume that the solution $y = \phi(t)$ satisfies the conditions of Corollary 6.2.2 for $n = 4$. The Laplace transform of the differential equation (25) is

$$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - Y(s) = 0.$$

Then, using the initial conditions (26) and solving for $Y(s)$, we have

$$Y(s) = \frac{s^2}{s^4 - 1}. \tag{27}$$

A partial fraction expansion of $Y(s)$ is

$$Y(s) = \frac{as + b}{s^2 - 1} + \frac{cs + d}{s^2 + 1},$$

and it follows that

$$(as + b)(s^2 + 1) + (cs + d)(s^2 - 1) = s^2 \tag{28}$$

for all s. By setting $s = 1$ and $s = -1$, respectively, in Eq. (28) we obtain the pair of equations

$$2(a + b) = 1, \qquad 2(-a + b) = 1,$$

and therefore $a = 0$ and $b = \frac{1}{2}$. If we set $s = 0$ in Eq. (28), then $b - d = 0$, so $d = \frac{1}{2}$. Finally, equating the coefficients of the cubic terms on each side of Eq. (28), we find that $a + c = 0$, so $c = 0$. Thus

$$Y(s) = \frac{1/2}{s^2 - 1} + \frac{1/2}{s^2 + 1}, \tag{29}$$

and from lines 7 and 5 of Table 6.2.1 the solution of the initial value problem (25), (26) is

$$y = \phi(t) = \frac{\sinh t + \sin t}{2}. \tag{30}$$

The most important elementary applications of the Laplace transform are in the study of mechanical vibrations and in the analysis of electric circuits; the governing equations were derived in Section 3.8. A vibrating spring–mass system has the equation of motion

$$m\frac{d^2u}{dt^2} + \gamma\frac{du}{dt} + ku = F(t), \tag{31}$$

where m is the mass, γ the damping coefficient, k the spring constant, and $F(t)$ the applied external force. The equation describing an electric circuit containing an inductance L, a resistance R, and a capacitance C (an LRC circuit) is

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t), \tag{32}$$

where $Q(t)$ is the charge on the capacitor and $E(t)$ is the applied voltage. In terms of the current $I(t) = dQ(t)/dt$ we can differentiate Eq. (32) and write

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}(t). \tag{33}$$

Suitable initial conditions on u, Q, or I must also be prescribed.

We have noted previously in Section 3.8 that Eq. (31) for the spring–mass system and Eq. (32) or (33) for the electric circuit are identical mathematically, differing only in the interpretation of the constants and variables appearing in them. There are other physical problems that also lead to the same differential equation. Thus, once the mathematical problem is solved, its solution can be interpreted in terms of whichever corresponding physical problem is of immediate interest.

In the problem lists following this and other sections in this chapter are numerous initial value problems for second order linear differential equations with constant coefficients. Many can be interpreted as models of particular physical systems, but usually we do not point this out explicitly.

PROBLEMS

In each of Problems 1 through 10 find the inverse Laplace transform of the given function.

1. $\dfrac{3}{s^2+4}$
2. $\dfrac{4}{(s-1)^3}$
3. $\dfrac{2}{s^2+3s-4}$
4. $\dfrac{3s}{s^2-s-6}$
5. $\dfrac{2s+2}{s^2+2s+5}$
6. $\dfrac{2s-3}{s^2-4}$
7. $\dfrac{2s+1}{s^2-2s+2}$
8. $\dfrac{8s^2-4s+12}{s(s^2+4)}$
9. $\dfrac{1-2s}{s^2+4s+5}$
10. $\dfrac{2s-3}{s^2+2s+10}$

In each of Problems 11 through 23 use the Laplace transform to solve the given initial value problem.

11. $y''-y'-6y=0; \quad y(0)=1, \quad y'(0)=-1$
12. $y''+3y'+2y=0; \quad y(0)=1, \quad y'(0)=0$
13. $y''-2y'+2y=0; \quad y(0)=0, \quad y'(0)=1$
14. $y''-4y'+4y=0; \quad y(0)=1, \quad y'(0)=1$
15. $y''-2y'-2y=0; \quad y(0)=2, \quad y'(0)=0$
16. $y''+2y'+5y=0; \quad y(0)=2, \quad y'(0)=-1$
17. $y^{\text{iv}}-4y'''+6y''-4y'+y=0; \quad y(0)=0, \quad y'(0)=1, \quad y''(0)=0, \quad y'''(0)=1$
18. $y^{\text{iv}}-y=0; \quad y(0)=1, \quad y'(0)=0, \quad y''(0)=1, \quad y'''(0)=0$
19. $y^{\text{iv}}-4y=0; \quad y(0)=1, \quad y'(0)=0, \quad y''(0)=-2, \quad y'''(0)=0$
20. $y''+\omega^2y=\cos 2t, \quad \omega^2\neq 4; \quad y(0)=1, \quad y'(0)=0$
21. $y''-2y'+2y=\cos t; \quad y(0)=1, \quad y'(0)=0$
22. $y''-2y'+2y=e^{-t}; \quad y(0)=0, \quad y'(0)=1$
23. $y''+2y'+y=4e^{-t}; \quad y(0)=2, \quad y'(0)=-1$

In each of Problems 24 through 26 find the Laplace transform $Y(s)=\mathcal{L}\{y\}$ of the solution of the given initial value problem. A method of determining the inverse transform is developed in Section 6.3.

24. $y''+4y=\begin{cases} 1, & 0\le t<\pi, \\ 0, & \pi\le t<\infty; \end{cases} \quad y(0)=1, \quad y'(0)=0$

25. $y''+y=\begin{cases} t, & 0\le t<1, \\ 0, & 1\le t<\infty; \end{cases} \quad y(0)=0, \quad y'(0)=0$

26. $y'' + 4y = \begin{cases} t, & 0 \le t < 1, \\ 1, & 1 \le t < \infty; \end{cases} \quad y(0) = 0, \quad y'(0) = 0$

27. The Laplace transforms of certain functions can be found conveniently from their Taylor series expansions.
(a) Using the Taylor series for $\sin t$,

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!},$$

and assuming that the Laplace transform of this series can be computed term by term, verify that

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}, \qquad s > 1.$$

(b) Let

$$f(t) = \begin{cases} (\sin t)/t, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

Find the Taylor series for f about $t = 0$. Assuming that the Laplace transform of this function can be computed term by term, verify that

$$\mathcal{L}\{f(t)\} = \arctan(1/s), \qquad s > 1.$$

(c) The Bessel function of the first kind of order zero J_0 has the Taylor series (see Section 5.8)

$$J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}.$$

Assuming that the following Laplace transforms can be computed term by term, verify that

$$\mathcal{L}\{J_0(t)\} = (s^2+1)^{-1/2}, \qquad s > 1,$$

and

$$\mathcal{L}\{J_0(\sqrt{t})\} = s^{-1}e^{-1/4s}, \qquad s > 0.$$

Problems 28 through 36 are concerned with differentiation of the Laplace transform.

28. Let

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt.$$

It is possible to show that as long as f satisfies the conditions of Theorem 6.1.2, it is legitimate to differentiate under the integral sign with respect to the parameter s when $s > a$.
(a) Show that $F'(s) = \mathcal{L}\{-tf(t)\}$.
(b) Show that $F^{(n)}(s) = \mathcal{L}\{(-t)^n f(t)\}$; hence differentiating the Laplace transform corresponds to multiplying the original function by $-t$.

In each of Problems 29 through 34 use the result of Problem 28 to find the Laplace transform of the given function; a and b are real numbers and n is a positive integer.

29. te^{at}
30. $t^2 \sin bt$
31. t^n
32. $t^n e^{at}$
33. $te^{at} \sin bt$
34. $te^{at} \cos bt$
35. Consider Bessel's equation of order zero

$$ty'' + y' + ty = 0.$$

Recall from Section 5.4 that $t = 0$ is a regular singular point for this equation, and therefore solutions may become unbounded as $t \to 0$. However, let us try to determine whether there are any solutions that remain finite at $t = 0$ and have finite derivatives there. Assuming that there is such a solution $y = \phi(t)$, let $Y(s) = \mathcal{L}\{\phi(t)\}$.
(a) Show that $Y(s)$ satisfies

$$(1 + s^2)Y'(s) + sY(s) = 0.$$

(b) Show that $Y(s) = c(1 + s^2)^{-1/2}$, where c is an arbitrary constant.
(c) Expanding $(1 + s^2)^{-1/2}$ in a binomial series valid for $s > 1$ and assuming that it is permissible to take the inverse transform term by term, show that

$$y = c\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2} = cJ_0(t),$$

where J_0 is the Bessel function of the first kind of order zero. Note that $J_0(0) = 1$, and that J_0 has finite derivatives of all orders at $t = 0$. It was shown in Section 5.8 that the second solution of this equation becomes unbounded as $t \to 0$.

36. For each of the following initial value problems use the results of Problem 28 to find the differential equation satisfied by $Y(s) = \mathcal{L}\{\phi(t)\}$, where $y = \phi(t)$ is the solution of the given initial value problem.
(a) $y'' - ty = 0; \quad y(0) = 1, \quad y'(0) = 0$ (Airy's equation)
(b) $(1 - t^2)y'' - 2ty' + \alpha(\alpha + 1)y = 0; \quad y(0) = 0, \quad y'(0) = 1$ (Legendre's equation)

Note that the differential equation for $Y(s)$ is of first order in part (a), but of second order in part (b). This is due to the fact that t appears at most to the first power in the equation of part (a), whereas it appears to the second power in that of part (b). This illustrates that the Laplace transform is not often useful in solving differential equations with variable coefficients, unless all the coefficients are at most linear functions of the independent variable.

37. Suppose that

$$g(t) = \int_0^t f(\tau)\,d\tau.$$

If $G(s)$ and $F(s)$ are the Laplace transforms of $g(t)$ and $f(t)$, respectively, show that

$$G(s) = F(s)/s.$$

38. In this problem we show how a general partial fraction expansion can be used to calculate many inverse Laplace transforms. Suppose that

$$F(s) = P(s)/Q(s),$$

where $Q(s)$ is a polynomial of degree n with distinct zeros $r_1, \dots, r_n$ and $P(s)$ is a polynomial of degree less than n. In this case it is possible to show that $P(s)/Q(s)$ has a partial fraction expansion of the form

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s - r_1} + \cdots + \frac{A_n}{s - r_n}, \tag{i}$$

where the coefficients $A_1, \dots, A_n$ must be determined.
(a) Show that

$$A_k = P(r_k)/Q'(r_k), \qquad k = 1, \dots, n. \tag{ii}$$

Hint: One way to do this is to multiply Eq. (i) by $s - r_k$ and then to take the limit as $s \to r_k$.

(b) Show that

$$\mathcal{L}^{-1}\{F(s)\} = \sum_{k=1}^{n} \frac{P(r_k)}{Q'(r_k)} e^{r_k t}. \tag{iii}$$

6.3 Step Functions

In Section 6.2 we outlined the general procedure involved in solving initial value problems by means of the Laplace transform. Some of the most interesting elementary applications of the transform method occur in the solution of linear differential equations with discontinuous or impulsive forcing functions. Equations of this type frequently arise in the analysis of the flow of current in electric circuits or the vibrations of mechanical systems. In this section and the following ones we develop some additional properties of the Laplace transform that are useful in the solution of such problems. Unless a specific statement is made to the contrary, all functions appearing below will be assumed to be piecewise continuous and of exponential order, so that their Laplace transforms exist, at least for s sufficiently large.

To deal effectively with functions having jump discontinuities, it is very helpful to introduce a function known as the **unit step function**, or **Heaviside function**. This function will be denoted by u_c, and is defined by

$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c, \end{cases} \qquad c \geq 0. \tag{1}$$

The graph of $y = u_c(t)$ is shown in Figure 6.3.1. The step can also be negative. For instance, Figure 6.3.2 shows the graph $y = 1 - u_c(t)$.

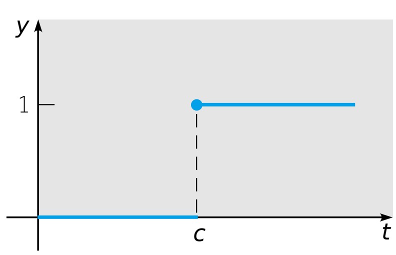

FIGURE 6.3.1 Graph of $y = u_c(t)$.

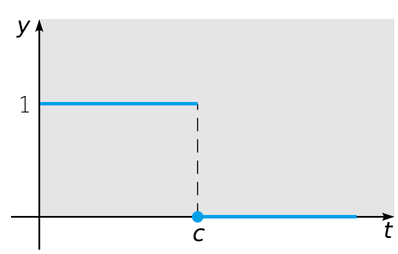

FIGURE 6.3.2 Graph of $y = 1 - u_c(t)$.

EXAMPLE 1

Sketch the graph of $y = h(t)$, where

$$h(t) = u_\pi(t) - u_{2\pi}(t), \qquad t \geq 0.$$

From the definition of $u_c(t)$ in Eq. (1) we have

$$h(t) = \begin{cases} 0 - 0 = 0, & 0 \leq t < \pi, \\ 1 - 0 = 1, & \pi \leq t < 2\pi, \\ 1 - 1 = 0, & 2\pi \leq t < \infty. \end{cases}$$

Thus the equation $y = h(t)$ has the graph shown in Figure 6.3.3. This function can be thought of as a rectangular pulse.

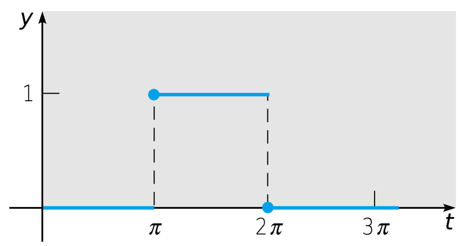

FIGURE 6.3.3 Graph of $y = u_\pi(t) - u_{2\pi}(t)$.

The Laplace transform of u_c is easily determined:

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st}u_c(t)\,dt = \int_c^\infty e^{-st}\,dt$$
$$= \frac{e^{-cs}}{s}, \qquad s > 0. \tag{2}$$

For a given function f, defined for $t \geq 0$, we will often want to consider the related function g defined by

$$y = g(t) = \begin{cases} 0, & t < c, \\ f(t-c), & t \geq c, \end{cases}$$

which represents a translation of f a distance c in the positive t direction; see Figure 6.3.4. In terms of the unit step function we can write $g(t)$ in the convenient form

$$g(t) = u_c(t)f(t-c).$$

The unit step function is particularly important in transform use because of the following relation between the transform of $f(t)$ and that of its translation $u_c(t)f(t-c)$.

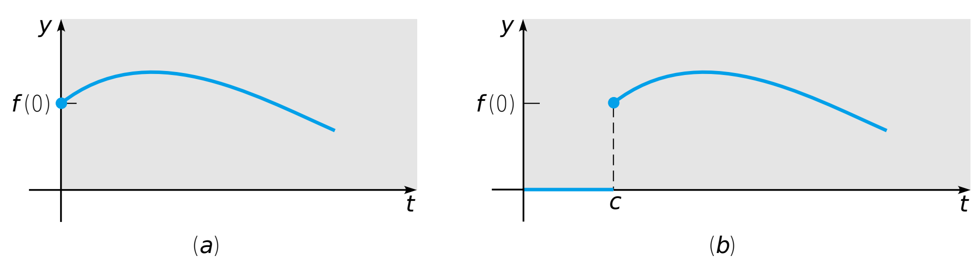

FIGURE 6.3.4 A translation of the given function. (a) $y = f(t)$; (b) $y = u_c(t)f(t-c)$.

Theorem 6.3.1 If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if c is a positive constant, then

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s), \qquad s > a. \tag{3}$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$u_c(t) f(t-c) = \mathcal{L}^{-1}\{e^{-cs} F(s)\}. \tag{4}$$

Theorem 6.3.1 simply states that the translation of $f(t)$ a distance c in the positive t direction corresponds to the multiplication of $F(s)$ by e^{-cs}. To prove Theorem 6.3.1 it is sufficient to compute the transform of $u_c(t) f(t-c)$:

$$\begin{aligned} \mathcal{L}\{u_c(t) f(t-c)\} &= \int_0^\infty e^{-st} u_c(t) f(t-c)\, dt \\ &= \int_c^\infty e^{-st} f(t-c)\, dt. \end{aligned}$$

Introducing a new integration variable $\xi = t - c$, we have

$$\begin{aligned} \mathcal{L}\{u_c(t) f(t-c)\} &= \int_0^\infty e^{-(\xi+c)s} f(\xi)\, d\xi = e^{-cs} \int_0^\infty e^{-s\xi} f(\xi)\, d\xi \\ &= e^{-cs} F(s). \end{aligned}$$

Thus Eq. (3) is established; Eq. (4) follows by taking the inverse transform of both sides of Eq. (3).

A simple example of this theorem occurs if we take $f(t) = 1$. Recalling that $\mathcal{L}\{1\} = 1/s$, we immediately have from Eq. (3) that $\mathcal{L}\{u_c(t)\} = e^{-cs}/s$. This result agrees with that of Eq. (2). Examples 2 and 3 illustrate further how Theorem 6.3.1 can be used in the calculation of transforms and inverse transforms.

EXAMPLE 2

If the function f is defined by

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi/4, \\ \sin t + \cos(t - \pi/4), & t \ge \pi/4, \end{cases}$$

find $\mathcal{L}\{f(t)\}$. The graph of $y = f(t)$ is shown in Figure 6.3.5.

Note that $f(t) = \sin t + g(t)$, where

$$g(t) = \begin{cases} 0, & t < \pi/4, \\ \cos(t - \pi/4), & t \ge \pi/4. \end{cases}$$

Thus

$$g(t) = u_{\pi/4}(t) \cos(t - \pi/4),$$

and

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{\sin t\} + \mathcal{L}\{u_{\pi/4}(t) \cos(t - \pi/4)\} \\ &= \mathcal{L}\{\sin t\} + e^{-\pi s/4} \mathcal{L}\{\cos t\}. \end{aligned}$$

Introducing the transforms of $\sin t$ and $\cos t$, we obtain

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2+1} + e^{-\pi s/4} \frac{s}{s^2+1} = \frac{1 + s e^{-\pi s/4}}{s^2+1}.$$

You should compare this method with the calculation of $\mathcal{L}\{f(t)\}$ directly from the definition.

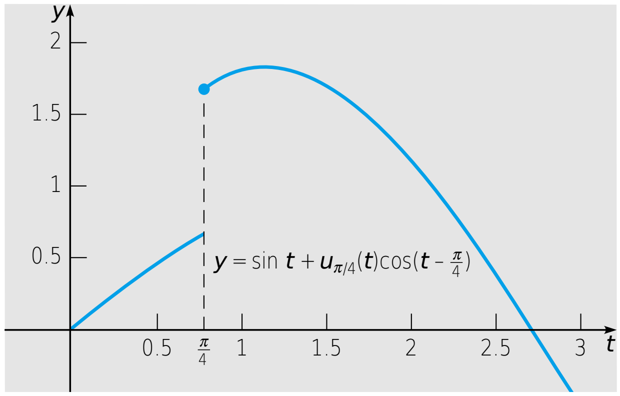

FIGURE 6.3.5 Graph of the function in Example 2.

EXAMPLE 3

Find the inverse transform of

$$F(s) = \frac{1 - e^{-2s}}{s^2}.$$

From the linearity of the inverse transform we have

$$\begin{aligned} f(t) = \mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} \\ &= t - u_2(t)(t-2). \end{aligned}$$

The function f may also be written as

$$f(t) = \begin{cases} t, & 0 \le t < 2, \\ 2, & t \ge 2. \end{cases}$$

The following theorem contains another very useful property of Laplace transforms that is somewhat analogous to that given in Theorem 6.3.1.

Theorem 6.3.2 If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \ge 0$, and if c is a constant, then

$$\mathcal{L}\{e^{ct} f(t)\} = F(s-c), \qquad s > a + c. \tag{5}$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$e^{ct} f(t) = \mathcal{L}^{-1}\{F(s-c)\}. \tag{6}$$

According to Theorem 6.3.2, multiplication of $f(t)$ by e^{ct} results in a translation of the transform $F(s)$ a distance c in the positive s direction, and conversely. The proof of this theorem requires merely the evaluation of $\mathcal{L}\{e^{ct} f(t)\}$. Thus

$$\begin{aligned} \mathcal{L}\{e^{ct} f(t)\} &= \int_0^\infty e^{-st} e^{ct} f(t)\, dt = \int_0^\infty e^{-(s-c)t} f(t)\, dt \\ &= F(s-c), \end{aligned}$$

which is Eq. (5). The restriction $s > a + c$ follows from the observation that, according to hypothesis (ii) of Theorem 6.1.2, $|f(t)| \le Ke^{at}$; hence $|e^{ct}f(t)| \le Ke^{(a+c)t}$. Equation (6) follows by taking the inverse transform of Eq. (5), and the proof is complete.

The principal application of Theorem 6.3.2 is in the evaluation of certain inverse transforms, as illustrated by Example 4.

EXAMPLE 4

Find the inverse transform of

$$G(s) = \frac{1}{s^2 - 4s + 5}.$$

By completing the square in the denominator we can write

$$G(s) = \frac{1}{(s-2)^2 + 1} = F(s-2),$$

where $F(s) = (s^2 + 1)^{-1}$. Since $\mathcal{L}^{-1}\{F(s)\} = \sin t$, it follows from Theorem 6.3.2 that

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = e^{2t} \sin t.$$

The results of this section are often useful in solving differential equations, particularly those having discontinuous forcing functions. The next section is devoted to examples illustrating this fact.

PROBLEMS

In each of Problems 1 through 6 sketch the graph of the given function on the interval $t \ge 0$.

1. $u_1(t) + 2u_3(t) - 6u_4(t)$
2. $(t-3)u_2(t) - (t-2)u_3(t)$
3. $f(t-\pi)u_\pi(t)$, where $f(t) = t^2$
4. $f(t-3)u_3(t)$, where $f(t) = \sin t$
5. $f(t-1)u_2(t)$, where $f(t) = 2t$
6. $f(t) = (t-1)u_1(t) - 2(t-2)u_2(t) + (t-3)u_3(t)$

In each of Problems 7 through 12 find the Laplace transform of the given function.

7. $f(t) = \begin{cases} 0, & t < 2 \\ (t-2)^2, & t \ge 2 \end{cases}$
8. $f(t) = \begin{cases} 0, & t < 1 \\ t^2 - 2t + 2, & t \ge 1 \end{cases}$
9. $f(t) = \begin{cases} 0, & t < \pi \\ t - \pi, & \pi \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$
10. $f(t) = u_1(t) + 2u_3(t) - 6u_4(t)$
11. $f(t) = (t-3)u_2(t) - (t-2)u_3(t)$
12. $f(t) = t - u_1(t)(t-1), \quad t \ge 0$

In each of Problems 13 through 18 find the inverse Laplace transform of the given function.

13. $F(s) = \dfrac{3!}{(s-2)^4}$
14. $F(s) = \dfrac{e^{-2s}}{s^2 + s - 2}$
15. $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2 - 2s + 2}$
16. $F(s) = \dfrac{2e^{-2s}}{s^2 - 4}$
17. $F(s) = \dfrac{(s-2)e^{-s}}{s^2 - 4s + 3}$
18. $F(s) = \dfrac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}$

19. Suppose that $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$.
    (a) Show that if c is a positive constant, then

$$\mathcal{L}\{f(ct)\} = \frac{1}{c}F\left(\frac{s}{c}\right), \qquad s > ca.$$

    (b) Show that if k is a positive constant, then

$$\mathcal{L}^{-1}\{F(ks)\} = \frac{1}{k}f\left(\frac{t}{k}\right).$$

    (c) Show that if a and b are constants with $a > 0$, then

$$\mathcal{L}^{-1}\{F(as + b)\} = \frac{1}{a}e^{-bt/a}f\left(\frac{t}{a}\right).$$

In each of Problems 20 through 23 use the results of Problem 19 to find the inverse Laplace transform of the given function.

20. $F(s) = \dfrac{2^{n+1}n!}{s^{n+1}}$

21. $F(s) = \dfrac{2s + 1}{4s^2 + 4s + 5}$

22. $F(s) = \dfrac{1}{9s^2 - 12s + 3}$

23. $F(s) = \dfrac{e^2e^{-4s}}{2s - 1}$

In each of Problems 24 through 27 find the Laplace transform of the given function. In Problem 27 assume that term-by-term integration of the infinite series is permissible.

24. $f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & t \geq 1 \end{cases}$

25. $f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & 1 \leq t < 2 \\ 1, & 2 \leq t < 3 \\ 0, & t \geq 3 \end{cases}$

26. $f(t) = 1 - u_1(t) + \cdots + u_{2n}(t) - u_{2n+1}(t) = 1 + \displaystyle\sum_{k=1}^{2n+1}(-1)^k u_k(t)$

27. $f(t) = 1 + \displaystyle\sum_{k=1}^{\infty}(-1)^k u_k(t)$. See Figure 6.3.6.

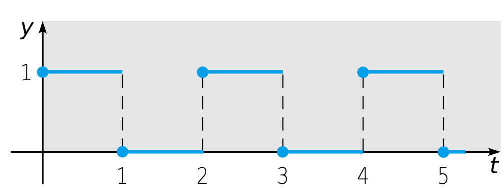

FIGURE 6.3.6 A square wave.

28. Let f satisfy $f(t + T) = f(t)$ for all $t \geq 0$ and for some fixed positive number T; f is said to be periodic with period T on $0 \leq t < \infty$. Show that

$$\mathcal{L}\{f(t)\} = \frac{\displaystyle\int_0^T e^{-st}f(t)\,dt}{1 - e^{-sT}}.$$

In each of Problems 29 through 32, use the result of Problem 28 to find the Laplace transform of the given function.

29. $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & 1 \le t < 2; \end{cases}$
$f(t+2) = f(t)$.
Compare with Problem 27.

30. $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ -1, & 1 \le t < 2; \end{cases}$
$f(t+2) = f(t)$.
See Figure 6.3.7.

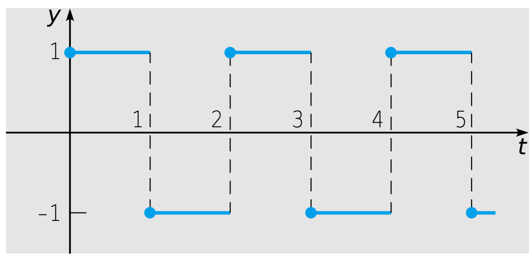

FIGURE 6.3.7 A square wave.

31. $f(t) = t, \quad 0 \le t < 1;$
$f(t+1) = f(t)$.
See Figure 6.3.8.

32. $f(t) = \sin t, \quad 0 \le t < \pi;$
$f(t+\pi) = f(t)$.
See Figure 6.3.9.

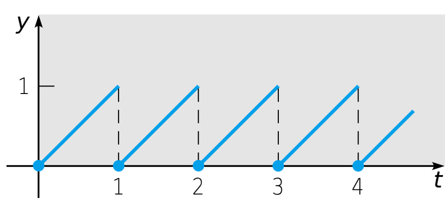

FIGURE 6.3.8 A sawtooth wave.

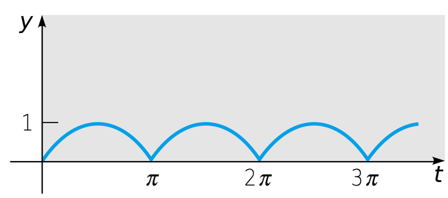

FIGURE 6.3.9 A rectified sine wave.

33. (a) If $f(t) = 1 - u_1(t)$, find $\mathcal{L}\{f(t)\}$; compare with Problem 24. Sketch the graph of $y = f(t)$.
(b) Let $g(t) = \int_0^t f(\xi)\,d\xi$, where the function f is defined in part (a). Sketch the graph of $y = g(t)$ and find $\mathcal{L}\{g(t)\}$.
(c) Let $h(t) = g(t) - u_1(t)g(t-1)$, where g is defined in part (b). Sketch the graph of $y = h(t)$ and find $\mathcal{L}\{h(t)\}$.

34. Consider the function p defined by

$$p(t) = \begin{cases} t, & 0 \le t < 1, \\ 2 - t, & 1 \le t < 2; \end{cases} \qquad p(t+2) = p(t).$$

(a) Sketch the graph of $y = p(t)$.
(b) Find $\mathcal{L}\{p(t)\}$ by noting that p is the periodic extension of the function h in Problem 33(c) and then using the result of Problem 28.

(c) Find $\mathcal{L}\{p(t)\}$ by noting that

$$p(t) = \int_0^t f(t)\, dt,$$

where f is the function in Problem 30, and then using Theorem 6.2.1.

6.4 Differential Equations with Discontinuous Forcing Functions

In this section we turn our attention to some examples in which the nonhomogeneous term, or forcing function, is discontinuous.

EXAMPLE 1

Find the solution of the differential equation

$$2y'' + y' + 2y = g(t), \tag{1}$$

where

$$g(t) = u_5(t) - u_{20}(t) = \begin{cases} 1, & 5 \le t < 20, \\ 0, & 0 \le t < 5 \quad \text{and} \quad t \ge 20. \end{cases} \tag{2}$$

Assume that the initial conditions are

$$y(0) = 0, \qquad y'(0) = 0. \tag{3}$$

This problem governs the charge on the capacitor in a simple electric circuit with a unit voltage pulse for $5 \le t < 20$. Alternatively, y may represent the response of a damped oscillator subject to the applied force $g(t)$.

The Laplace transform of Eq. (1) is

$$\begin{aligned} 2s^2Y(s) - 2sy(0) - 2y'(0) + sY(s) - y(0) + 2Y(s) &= \mathcal{L}\{u_5(t)\} - \mathcal{L}\{u_{20}(t)\} \\ &= (e^{-5s} - e^{-20s})/s. \end{aligned}$$

Introducing the initial values (3) and solving for $Y(s)$, we obtain

$$Y(s) = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)}. \tag{4}$$

To find $y = \phi(t)$ it is convenient to write $Y(s)$ as

$$Y(s) = (e^{-5s} - e^{-20s})H(s), \tag{5}$$

where

$$H(s) = 1/s(2s^2 + s + 2). \tag{6}$$

Then, if $h(t) = \mathcal{L}^{-1}\{H(s)\}$, we have

$$y = \phi(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20). \tag{7}$$

Observe that we have used Theorem 6.3.1 to write the inverse transforms of $e^{-5s}H(s)$ and $e^{-20s}H(s)$, respectively. Finally, to determine $h(t)$ we use the partial fraction expansion of $H(s)$:

$$H(s) = \frac{a}{s} + \frac{bs + c}{2s^2 + s + 2}. \tag{8}$$

Upon determining the coefficients we find that $a = \frac{1}{2}$, $b = -1$, and $c = -\frac{1}{2}$. Thus

$$\begin{aligned} H(s) &= \frac{1/2}{s} - \frac{s + \frac{1}{2}}{2s^2 + s + 2} \\ &= \frac{1/2}{s} - \left(\frac{1}{2}\right) \frac{(s + \frac{1}{4}) + \frac{1}{4}}{(s + \frac{1}{4})^2 + \frac{15}{16}}, \end{aligned} \tag{9}$$

so that, by referring to lines 9 and 10 of Table 6.2.1, we obtain

$$h(t) = \tfrac{1}{2} - \tfrac{1}{2}[e^{-t/4}\cos(\sqrt{15}\,t/4) + (\sqrt{15}/15)e^{-t/4}\sin(\sqrt{15}\,t/4)]. \tag{10}$$

In Figure 6.4.1 the graph of $y = \phi(t)$ from Eqs. (7) and (10) shows that the solution consists of three distinct parts. For $0 < t < 5$ the differential equation is

$$2y'' + y' + 2y = 0 \tag{11}$$

and the initial conditions are given by Eq. (3). Since the initial conditions impart no energy to the system, and since there is no external forcing, the system remains at rest; that is, $y = 0$ for $0 < t < 5$. This can be confirmed by solving Eq. (11) subject to the initial conditions (3). In particular, evaluating the solution and its derivative at $t = 5$, or more precisely, as t approaches 5 from below, we have

$$y(5) = 0, \qquad y'(5) = 0. \tag{12}$$

Once $t > 5$, the differential equation becomes

$$2y'' + y' + 2y = 1, \tag{13}$$

whose solution is the sum of a constant (the response to the constant forcing function) and a damped oscillation (the solution of the corresponding homogeneous equation). The plot in Figure 6.4.1 shows this behavior clearly for the interval $5 \le t \le 20$. An expression for this portion of the solution can be found by solving the differential equation (13) subject to the initial conditions (12). Finally, for $t > 20$ the differential equation becomes Eq. (11) again, and the initial conditions are obtained by evaluating the solution of Eqs. (13), (12) and its derivative at $t = 20$. These values are

$$y(20) \cong 0.50162, \qquad y'(20) \cong 0.01125. \tag{14}$$

The initial value problem (11), (14) contains no external forcing, so its solution is a damped oscillation about $y = 0$, as can be seen in Figure 6.4.1.

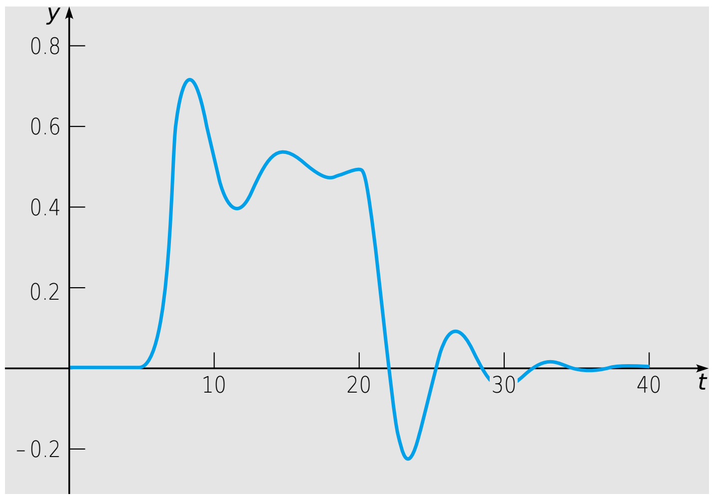

FIGURE 6.4.1 Solution of the initial value problem (1), (2), (3).

While it may be helpful to visualize the solution shown in Figure 6.4.1 as composed of solutions of three separate initial value problems in three separate intervals, it is somewhat tedious to find the solution by solving these separate problems. Laplace transform methods provide a much more convenient and elegant approach to this problem and to others having discontinuous forcing functions.

The effect of the discontinuity in the forcing function can be seen if we examine the solution $\phi(t)$ of Example 1 more closely. According to the existence and uniqueness Theorem 3.2.1 the solution ϕ and its first two derivatives are continuous except possibly at the points $t = 5$ and $t = 20$ where g is discontinuous. This can also be seen at once from Eq. (7). One can also show by direct computation from Eq. (7) that ϕ and ϕ' are continuous even at $t = 5$ and $t = 20$. However, if we calculate ϕ'', we find that

$$\lim_{t\to 5-} \phi''(t) = 0, \qquad \lim_{t\to 5+} \phi''(t) = 1/2.$$

Consequently, $\phi''(t)$ has a jump of $1/2$ at $t = 5$. In a similar way one can show that $\phi''(t)$ has a jump of $-1/2$ at $t = 20$. Thus the jump in the forcing term $g(t)$ at these points is balanced by a corresponding jump in the highest order term $2y''$ on the left side of the equation.

Consider now the general second order linear equation

$$y'' + p(t)y' + q(t)y = g(t), \tag{15}$$

where p and q are continuous on some interval $\alpha < t < \beta$, but g is only piecewise continuous there. If $y = \psi(t)$ is a solution of Eq. (15), then ψ and ψ' are continuous on $\alpha < t < \beta$, but ψ'' has jump discontinuities at the same points as g. Similar remarks apply to higher order equations; the highest derivative of the solution appearing in the differential equation has jump discontinuities at the same points as the forcing function, but the solution itself and its lower derivatives are continuous even at those points.

EXAMPLE 2

Describe the qualitative nature of the solution of the initial value problem

$$y'' + 4y = g(t), \tag{16}$$

$$y(0) = 0, \qquad y'(0) = 0, \tag{17}$$

where

$$g(t) = \begin{cases} 0, & 0 \le t < 5, \\ (t-5)/5, & 5 \le t < 10, \\ 1, & t \ge 10, \end{cases} \tag{18}$$

and then find the solution.

In this example the forcing function has the graph shown in Figure 6.4.2 and is known as ramp loading. It is relatively easy to identify the general form of the solution. For $t < 5$ the solution is simply $y = 0$. On the other hand, for $t > 10$ the solution has the form

$$y = c_1 \cos 2t + c_2 \sin 2t + 1/4. \tag{19}$$

The constant $1/4$ is a particular solution of the nonhomogeneous equation while the other two terms are the general solution of the corresponding homogeneous equation. Thus the solution (19) is a simple harmonic oscillation about $y = 1/4$. Similarly, in the intermediate range $5 < t < 10$, the solution is an oscillation about a certain linear function. In an engineering context, for example, we might be interested in knowing the amplitude of the eventual steady oscillation.

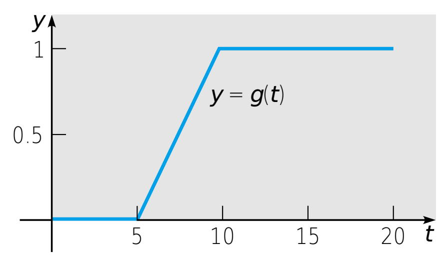

FIGURE 6.4.2 Ramp loading; $y = g(t)$ from Eq. (18).

To solve the problem it is convenient to write

$$g(t) = [u_5(t)(t-5) - u_{10}(t)(t-10)]/5, \tag{20}$$

as you may verify. Then we take the Laplace transform of the differential equation and use the initial conditions, thereby obtaining

$$(s^2+4)Y(s) = (e^{-5s} - e^{-10s})/5s^2,$$

or

$$Y(s) = (e^{-5s} - e^{-10s})H(s)/5, \tag{21}$$

where

$$H(s) = 1/s^2(s^2+4). \tag{22}$$

Thus the solution of the initial value problem (16), (17), (18) is

$$y = \phi(t) = [u_5(t)h(t-5) - u_{10}(t)h(t-10)]/5, \tag{23}$$

where $h(t)$ is the inverse transform of $H(s)$. The partial fraction expansion of $H(s)$ is

$$H(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2+4}, \tag{24}$$

and it then follows from lines 3 and 5 of Table 6.2.1 that

$$h(t) = \tfrac{1}{4}t - \tfrac{1}{8}\sin 2t. \tag{25}$$

The graph of $y = \phi(t)$ is shown in Figure 6.4.3. Observe that it has the qualitative form that we indicated earlier. To find the amplitude of the eventual steady oscillation it is sufficient to locate one of the maximum or minimum points for $t > 10$. Setting the derivative of the solution (23) equal to zero, we find that the first maximum is located approximately at (10.642, 0.2979), so the amplitude of the oscillation is approximately 0.0479.

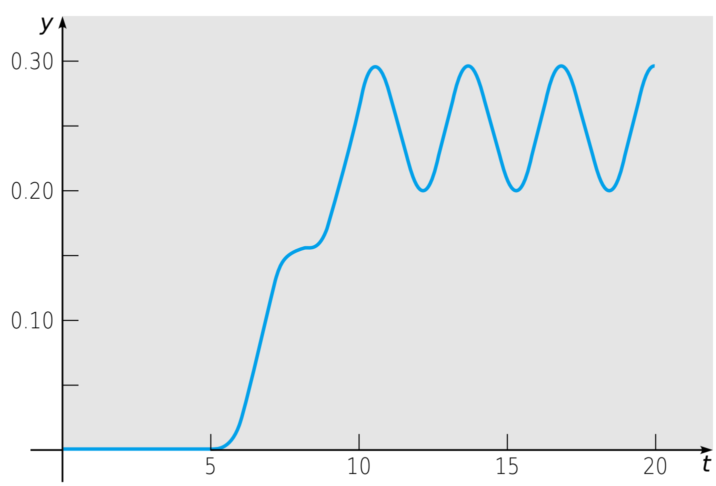

FIGURE 6.4.3 Solution of the initial value problem (16), (17), (18).

Note that in this example the forcing function g is continuous but g' is discontinuous at $t = 5$ and $t = 10$. It follows that the solution ϕ and its first two derivatives are continuous everywhere, but ϕ''' has discontinuities at $t = 5$ and at $t = 10$ that match the discontinuities in g' at those points.

PROBLEMS

In each of Problems 1 through 13 find the solution of the given initial value problem. Draw the graphs of the solution and of the forcing function; explain how they are related.

1. $y'' + y = f(t); \quad y(0) = 0, \quad y'(0) = 1; \quad f(t) = \begin{cases} 1, & 0 \le t < \pi/2 \\ 0, & \pi/2 \le t < \infty \end{cases}$

2. $y'' + 2y' + 2y = h(t); \quad y(0) = 0, \quad y'(0) = 1; \quad h(t) = \begin{cases} 1, & \pi \le t < 2\pi \\ 0, & 0 \le t < \pi \text{ and } t \ge 2\pi \end{cases}$

3. $y'' + 4y = \sin t - u_{2\pi}(t)\sin(t - 2\pi); \qquad y(0) = 0, \quad y'(0) = 0$
4. $y'' + 4y = \sin t + u_{\pi}(t)\sin(t - \pi); \qquad y(0) = 0, \quad y'(0) = 0$
5. $y'' + 3y' + 2y = f(t); \qquad y(0) = 0, \quad y'(0) = 0; \qquad f(t) = \begin{cases} 1, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$
6. $y'' + 3y' + 2y = u_2(t); \qquad y(0) = 0, \quad y'(0) = 1$
7. $y'' + y = u_{3\pi}(t); \qquad y(0) = 1, \quad y'(0) = 0$
8. $y'' + y' + \frac{5}{4}y = t - u_{\pi/2}(t)(t - \pi/2); \qquad y(0) = 0, \quad y'(0) = 0$
9. $y'' + y = g(t); \qquad y(0) = 0, \quad y'(0) = 1; \qquad g(t) = \begin{cases} t/2, & 0 \le t < 6 \\ 3, & t \ge 6 \end{cases}$
10. $y'' + y' + \frac{5}{4}y = g(t); \qquad y(0) = 0, \quad y'(0) = 0; \qquad g(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$
11. $y'' + 4y = u_{\pi}(t) - u_{3\pi}(t); \qquad y(0) = 0, \quad y'(0) = 0$
12. $y^{\text{iv}} - y = u_1(t) - u_2(t); \qquad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$
13. $y^{\text{iv}} + 5y'' + 4y = 1 - u_{\pi}(t); \qquad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$
14. Find an expression involving $u_c(t)$ for a function f that ramps up from zero at $t = t_0$ to the value h at $t = t_0 + k$.
15. Find an expression involving $u_c(t)$ for a function g that ramps up from zero at $t = t_0$ to the value h at $t = t_0 + k$ and then ramps back down to zero at $t = t_0 + 2k$.
16. A certain spring–mass system satisfies the initial value problem

$$u'' + \tfrac{1}{4}u' + u = kg(t), \qquad u(0) = 0, \quad u'(0) = 0,$$

where $g(t) = u_{3/2}(t) - u_{5/2}(t)$ and $k > 0$ is a parameter.
(a) Sketch the graph of $g(t)$. Observe that it is a pulse of unit magnitude extending over one time unit.
(b) Solve the initial value problem.
(c) Plot the solution for $k = 1/2$, $k = 1$, and $k = 2$. Describe the principal features of the solution and how they depend on k.
(d) Find, to two decimal places, the smallest value of k for which the solution $u(t)$ reaches the value 2.
(e) Suppose $k = 2$. Find the time τ after which $|u(t)| < 0.1$ for all $t > \tau$.
17. Modify the problem in Example 2 in the text by replacing the given forcing function $g(t)$ by

$$f(t) = [u_5(t)(t - 5) - u_{5+k}(t)(t - 5 - k)]/k.$$

(a) Sketch the graph of $f(t)$ and describe how it depends on k. For what value of k is $f(t)$ identical to $g(t)$ in the example?
(b) Solve the initial value problem

$$y'' + 4y = f(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

(c) The solution in part (b) depends on k, but for sufficiently large t the solution is always a simple harmonic oscillation about $y = 1/4$. Try to decide how the amplitude of this eventual oscillation depends on k. Then confirm your conclusion by plotting the solution for a few different values of k.
18. Consider the initial value problem

$$y'' + \tfrac{1}{3}y' + 4y = f_k(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where

$$f_k(t) = \begin{cases} 1/2k, & 4 - k \le t < 4 + k \\ 0, & 0 \le t < 4 - k \quad \text{and} \quad t \ge 4 + k \end{cases}$$

and $0 < k < 4$.

(a) Sketch the graph of $f_k(t)$. Observe that the area under the graph is independent of k. If $f_k(t)$ represents a force, this means that the product of the magnitude of the force and the time interval during which it acts does not depend on k.
(b) Write $f_k(t)$ in terms of the unit step function and then solve the given initial value problem.
(c) Plot the solution for $k = 2$, $k = 1$, and $k = \frac{1}{2}$. Describe how the solution depends on k.

Resonance and Beats. In Section 3.9 we observed that an undamped harmonic oscillator (such as a spring–mass system), with a sinusoidal forcing term experiences resonance if the frequency of the forcing term is the same as the natural frequency. If the forcing frequency is slightly different from the natural frequency, then the system exhibits a beat. In Problems 19 through 23 we explore the effect of some nonsinusoidal periodic forcing functions.

19. Consider the initial value problem

$$y'' + y = f(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where

$$f(t) = u_0(t) + 2\sum_{k=1}^{n}(-1)^k u_{k\pi}(t).$$

(a) Draw the graph of $f(t)$ on an interval such as $0 \le t \le 6\pi$.
(b) Find the solution of the initial value problem.
(c) Let $n = 15$ and plot the graph of the solution for $0 \le t \le 60$. Describe the solution and explain why it behaves as it does.
(d) Investigate how the solution changes as n increases. What happens as $n \to \infty$?

20. Consider the initial value problem

$$y'' + 0.1y' + y = f(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where $f(t)$ is the same as in Problem 19.
(a) Plot the graph of the solution. Use a large enough value of n and a long enough t-interval so that the transient part of the solution has become negligible and the steady state is clearly shown.
(b) Estimate the amplitude and frequency of the steady-state part of the solution.
(c) Compare the results of part (b) with those from Section 3.9 for a sinusoidally forced oscillator.

21. Consider the initial value problem

$$y'' + y = g(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where

$$g(t) = u_0(t) + \sum_{k=1}^{n}(-1)^k u_{k\pi}(t).$$

(a) Draw the graph of $g(t)$ on an interval such as $0 \le t \le 6\pi$. Compare the graph with that of $f(t)$ in Problem 19(a).
(b) Find the solution of the initial value problem.
(c) Let $n = 15$ and plot the graph of the solution for $0 \le t \le 60$. Describe the solution and explain why it behaves as it does. Compare it with the solution of Problem 19.
(d) Investigate how the solution changes as n increases. What happens as $n \to \infty$?

22. Consider the initial value problem

$$y'' + 0.1y' + y = g(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where $g(t)$ is the same as in Problem 21.

(a) Plot the graph of the solution. Use a large enough value of n and a long enough t-interval so that the transient part of the solution has become negligible and the steady state is clearly shown.
(b) Estimate the amplitude and frequency of the steady-state part of the solution.
(c) Compare the results of part (b) with those from Problem 20 and from Section 3.9 for a sinusoidally forced oscillator.

23. Consider the initial value problem

$$y'' + y = h(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where

$$f(t) = u_0(t) + 2\sum_{k=1}^{n}(-1)^k u_{11k/4}(t).$$

Observe that this problem is identical to Problem 19 except that the frequency of the forcing term has been increased somewhat.
(a) Find the solution of this initial value problem.
(b) Let $n \geq 33$ and plot the solution for $0 \leq t \leq 90$ or longer. Your plot should show a clearly recognizable beat.
(c) From the graph in part (b) estimate the "slow period" and the "fast period" for this oscillator.
(d) For a sinusoidally forced oscillator in Section 3.9 it was shown that the "slow frequency" is given by $|\omega - \omega_0|/2$, where ω_0 is the natural frequency of the system and ω is the forcing frequency. Similarly the "fast frequency" is $(\omega + \omega_0)/2$. Use these expressions to calculate the "fast period" and the "slow period" for the oscillator in this problem. How well do the results compare with your estimates from part (c)?

6.5 Impulse Functions

In some applications it is necessary to deal with phenomena of an impulsive nature, for example, voltages or forces of large magnitude that act over very short time intervals. Such problems often lead to differential equations of the form

$$ay'' + by' + cy = g(t), \tag{1}$$

where $g(t)$ is large during a short interval $t_0 - \tau < t < t_0 + \tau$ and is otherwise zero.

The integral $I(\tau)$, defined by

$$I(\tau) = \int_{t_0-\tau}^{t_0+\tau} g(t)\, dt, \tag{2}$$

or, since $g(t) = 0$ outside of the interval $(t_0 - \tau, t_0 + \tau)$,

$$I(\tau) = \int_{-\infty}^{\infty} g(t)\, dt, \tag{3}$$

is a measure of the strength of the forcing function. In a mechanical system, where $g(t)$ is a force, $I(\tau)$ is the total **impulse** of the force $g(t)$ over the time interval $(t_0 - \tau, t_0 + \tau)$. Similarly, if y is the current in an electric circuit and $g(t)$ is the time derivative of the voltage, then $I(\tau)$ represents the total voltage impressed on the circuit during the interval $(t_0 - \tau, t_0 + \tau)$.

In particular, let us suppose that t_0 is zero, and that $g(t)$ is given by

$$g(t) = d_\tau(t) = \begin{cases} 1/2\tau, & -\tau < t < \tau, \\ 0, & t \le -\tau \quad \text{or} \quad t \ge \tau, \end{cases} \tag{4}$$

where τ is a small positive constant (see Figure 6.5.1). According to Eq. (2) or (3) it follows immediately that in this case $I(\tau) = 1$ independent of the value of τ, as long as $\tau \neq 0$. Now let us idealize the forcing function d_τ by prescribing it to act over shorter and shorter time intervals; that is, we require that $\tau \to 0$, as indicated in Figure 6.5.2. As a result of this limiting operation we obtain

$$\lim_{\tau \to 0} d_\tau(t) = 0, \qquad t \neq 0. \tag{5}$$

Further, since $I(\tau) = 1$ for each $\tau \neq 0$, it follows that

$$\lim_{\tau \to 0} I(\tau) = 1. \tag{6}$$

Equations (5) and (6) can be used to define an idealized **unit impulse function** δ, which imparts an impulse of magnitude one at $t = 0$, but is zero for all values of t other than zero. That is, the "function" δ is defined to have the properties

$$\delta(t) = 0, \qquad t \neq 0; \tag{7}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1. \tag{8}$$

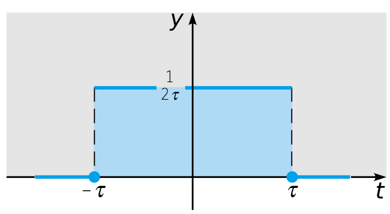

FIGURE 6.5.1 Graph of $y = d_\tau(t)$.

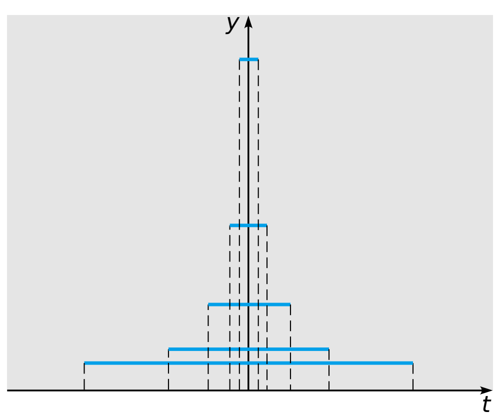

FIGURE 6.5.2 Graphs of $y = d_\tau(t)$ as $\tau \to 0$.

There is no ordinary function of the kind studied in elementary calculus that satisfies both of Eqs. (7) and (8). The "function" δ, defined by those equations, is an example of what are known as generalized functions and is usually called the Dirac[2] **delta function**. Since $\delta(t)$ corresponds to a unit impulse at $t = 0$, a unit impulse at an arbitrary point $t = t_0$ is given by $\delta(t - t_0)$. From Eqs. (7) and (8) it follows that

$$\delta(t - t_0) = 0, \qquad t \neq t_0; \tag{9}$$

$$\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1. \tag{10}$$

The delta function does not satisfy the conditions of Theorem 6.1.2, but its Laplace transform can nevertheless be formally defined. Since $\delta(t)$ is defined as the limit of $d_\tau(t)$ as $\tau \to 0$, it is natural to define the Laplace transform of δ as a similar limit of the transform of d_τ. In particular, we will assume that $t_0 > 0$, and define $\mathcal{L}\{\delta(t - t_0)\}$ by the equation

$$\mathcal{L}\{\delta(t - t_0)\} = \lim_{\tau \to 0} \mathcal{L}\{d_\tau(t - t_0)\}. \tag{11}$$

To evaluate the limit in Eq. (11) we first observe that if $\tau < t_0$, which must eventually be the case as $\tau \to 0$, then $t_0 - \tau > 0$. Since $d_\tau(t - t_0)$ is nonzero only in the interval from $t_0 - \tau$ to $t_0 + \tau$, we have

$$\begin{aligned} \mathcal{L}\{d_\tau(t - t_0)\} &= \int_0^\infty e^{-st} d_\tau(t - t_0)\, dt \\ &= \int_{t_0-\tau}^{t_0+\tau} e^{-st} d_\tau(t - t_0)\, dt. \end{aligned}$$

Substituting for $d_\tau(t - t_0)$ from Eq. (4), we obtain

$$\begin{aligned} \mathcal{L}\{d_\tau(t - t_0)\} &= \frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} e^{-st}\, dt = -\frac{1}{2s\tau} e^{-st}\Big|_{t=t_0-\tau}^{t=t_0+\tau} \\ &= \frac{1}{2s\tau} e^{-st_0}(e^{s\tau} - e^{-s\tau}) \end{aligned}$$

or

$$\mathcal{L}\{d_\tau(t - t_0)\} = \frac{\sinh s\tau}{s\tau} e^{-st_0}. \tag{12}$$

The quotient $(\sinh s\tau)/s\tau$ is indeterminate as $\tau \to 0$, but its limit can be evaluated by L'Hospital's rule. We obtain

$$\lim_{\tau \to 0} \frac{\sinh s\tau}{s\tau} = \lim_{\tau \to 0} \frac{s \cosh s\tau}{s} = 1.$$

Then from Eq. (11) it follows that

$$\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}. \tag{13}$$

[2]Paul A. M. Dirac (1902–1984), English mathematical physicist, received his Ph.D. from Cambridge in 1926 and was professor of mathematics there until 1969. He was awarded the Nobel prize in 1933 (with Erwin Schrödinger) for fundamental work in quantum mechanics. His most celebrated result was the relativistic equation for the electron, published in 1928. From this equation he predicted the existence of an "anti-electron," or positron, which was first observed in 1932. Following his retirement from Cambridge, Dirac moved to the United States and held a research professorship at Florida State University.

Equation (13) defines $\mathcal{L}\{\delta(t-t_0)\}$ for any $t_0 > 0$. We extend this result, to allow t_0 to be zero, by letting $t_0 \to 0$ on the right side of Eq. (13); thus

$$\mathcal{L}\{\delta(t)\} = \lim_{t_0 \to 0} e^{-st_0} = 1. \tag{14}$$

In a similar way it is possible to define the integral of the product of the delta function and any continuous function f. We have

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\,dt = \lim_{\tau \to 0} \int_{-\infty}^{\infty} d_\tau(t-t_0) f(t)\,dt. \tag{15}$$

Using the definition (4) of $d_\tau(t)$ and the mean value theorem for integrals, we find that

$$\begin{aligned}\int_{-\infty}^{\infty} d_\tau(t-t_0) f(t)\,dt &= \frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} f(t)\,dt \\ &= \frac{1}{2\tau} \cdot 2\tau \cdot f(t^*) = f(t^*),\end{aligned}$$

where $t_0 - \tau < t^* < t_0 + \tau$. Hence, $t^* \to t_0$ as $\tau \to 0$, and it follows from Eq. (15) that

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\,dt = f(t_0). \tag{16}$$

It is often convenient to introduce the delta function when working with impulse problems, and to operate formally on it as though it were a function of the ordinary kind. This is illustrated in the example below. It is important to realize, however, that the ultimate justification of such procedures must rest on a careful analysis of the limiting operations involved. Such a rigorous mathematical theory has been developed, but we do not discuss it here.

EXAMPLE 1

Find the solution of the initial value problem

$$2y'' + y' + 2y = \delta(t-5), \tag{17}$$

$$y(0) = 0, \qquad y'(0) = 0. \tag{18}$$

This initial value problem arises from the study of the same electrical circuit or mechanical oscillator as in Example 1 of Section 6.4. The only difference is in the forcing term.

To solve the given problem we take the Laplace transform of the differential equation and use the initial conditions, obtaining

$$(2s^2 + s + 2)Y(s) = e^{-5s}.$$

Thus

$$Y(s) = \frac{e^{-5s}}{2s^2 + s + 2} = \frac{e^{-5s}}{2}\,\frac{1}{(s+\frac{1}{4})^2 + \frac{15}{16}}. \tag{19}$$

By Theorem 6.3.2 or from line 9 of Table 6.2.1

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+\frac{1}{4})^2 + \frac{15}{16}}\right\} = \frac{4}{\sqrt{15}}\, e^{-t/4} \sin \frac{\sqrt{15}}{4}\,t. \tag{20}$$

Hence, by Theorem 6.3.1, we have

$$y = \mathcal{L}^{-1}\{Y(s)\} = \frac{2}{\sqrt{15}} u_5(t) e^{-(t-5)/4} \sin \frac{\sqrt{15}}{4}(t-5), \tag{21}$$

which is the formal solution of the given problem. It is also possible to write y in the form

$$y = \begin{cases} 0, & t < 5, \\ \frac{2}{\sqrt{15}} e^{-(t-5)/4} \sin \frac{\sqrt{15}}{4}(t-5), & t \geq 5. \end{cases} \tag{22}$$

The graph of Eq. (22) is shown in Figure 6.5.3. Since the initial conditions at $t = 0$ are homogeneous, and there is no external excitation until $t = 5$, there is no response in the interval $0 < t < 5$. The impulse at $t = 5$ produces a decaying oscillation that persists indefinitely. The response is continuous at $t = 5$ despite the singularity in the forcing function at that point. However, the first derivative of the solution has a jump discontinuity at $t = 5$ and the second derivative has an infinite discontinuity there. This is required by the differential equation (17), since a singularity on one side of the equation must be balanced by a corresponding singularity on the other side.

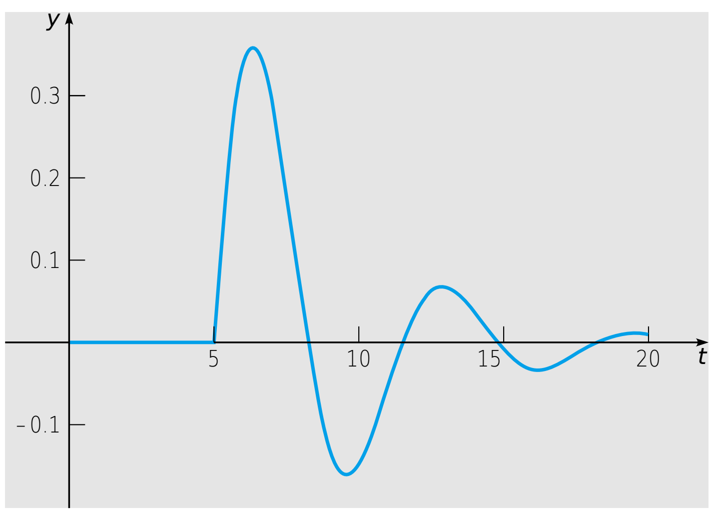

FIGURE 6.5.3 Solution of the initial value problem (17), (18).

PROBLEMS

In each of Problems 1 through 12 find the solution of the given initial value problem and draw its graph.

1. $y'' + 2y' + 2y = \delta(t - \pi);\quad y(0) = 1,\quad y'(0) = 0$
2. $y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi);\quad y(0) = 0,\quad y'(0) = 0$
3. $y'' + 3y' + 2y = \delta(t - 5) + u_{10}(t);\quad y(0) = 0,\quad y'(0) = 1/2$
4. $y'' - y = -20\delta(t - 3);\quad y(0) = 1,\quad y'(0) = 0$
5. $y'' + 2y' + 3y = \sin t + \delta(t - 3\pi);\quad y(0) = 0,\quad y'(0) = 0$
6. $y'' + 4y = \delta(t - 4\pi);\quad y(0) = 1/2,\quad y'(0) = 0$
7. $y'' + y = \delta(t - 2\pi)\cos t;\quad y(0) = 0,\quad y'(0) = 1$
8. $y'' + 4y = 2\delta(t - \pi/4);\quad y(0) = 0,\quad y'(0) = 0$

9. $y'' + y = u_{\pi/2}(t) + 3\delta(t - 3\pi/2) - u_{2\pi}(t); \qquad y(0) = 0, \quad y'(0) = 0$
10. $2y'' + y' + 4y = \delta(t - \pi/6)\sin t; \qquad y(0) = 0, \quad y'(0) = 0$
11. $y'' + 2y' + 2y = \cos t + \delta(t - \pi/2); \qquad y(0) = 0, \quad y'(0) = 0$
12. $y^{\text{iv}} - y = \delta(t - 1); \qquad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$
13. Consider again the system in Example 1 in the text in which an oscillation is excited by a unit impulse at $t = 5$. Suppose that it is desired to bring the system to rest again after exactly one cycle, that is, when the response first returns to equilibrium moving in the positive direction.
(a) Determine the impulse $k\delta(t - t_0)$ that should be applied to the system in order to accomplish this objective. Note that k is the magnitude of the impulse and t_0 is the time of its application.
(b) Solve the resulting initial value problem and plot its solution to confirm that it behaves in the specified manner.
14. Consider the initial value problem
$$y'' + \gamma y' + y = \delta(t - 1), \qquad y(0) = 0, \quad y'(0) = 0,$$
where γ is the damping coefficient (or resistance).
(a) Let $\gamma = \frac{1}{2}$. Find the solution of the initial value problem and plot its graph.
(b) Find the time t_1 at which the solution attains its maximum value. Also find the maximum value y_1 of the solution.
(c) Let $\gamma = \frac{1}{4}$ and repeat parts (a) and (b).
(d) Determine how t_1 and y_1 vary as γ decreases. What are the values of t_1 and y_1 when $\gamma = 0$?
15. Consider the initial value problem
$$y'' + \gamma y' + y = k\delta(t - 1), \qquad y(0) = 0, \quad y'(0) = 0,$$
where k is the magnitude of an impulse at $t = 1$ and γ is the damping coefficient (or resistance).
(a) Let $\gamma = \frac{1}{2}$. Find the value of k for which the response has a peak value of 2; call this value k_1.
(b) Repeat part (a) for $\gamma = \frac{1}{4}$.
(c) Determine how k_1 varies as γ decreases. What is the value of k_1 when $\gamma = 0$?
16. Consider the initial value problem
$$y'' + y = f_k(t), \qquad y(0) = 0, \quad y'(0) = 0,$$
where $f_k(t) = [u_{4-k}(t) - u_{4+k}(t)]/2k$ with $0 < k \le 1$.
(a) Find the solution $y = \phi(t, k)$ of the initial value problem.
(b) Calculate $\lim_{k\to 0} \phi(t, k)$ from the solution found in part (a).
(c) Observe that $\lim_{k\to 0} f_k(t) = \delta(t - 4)$. Find the solution $\phi_0(t)$ of the given initial value problem with $f_k(t)$ replaced by $\delta(t - 4)$. Is it true that $\phi_0(t) = \lim_{k\to 0} \phi(t, k)$?
(d) Plot $\phi(t, 1/2)$, $\phi(t, 1/4)$, and $\phi_0(t)$ on the same axes. Describe the relation between $\phi(t, k)$ and $\phi_0(t)$.

Problems 17 through 22 deal with the effect of a sequence of impulses on an undamped oscillator. Suppose that
$$y'' + y = f(t), \qquad y(0) = 0, \quad y'(0) = 0.$$
For each of the following choices for $f(t)$:

(a) Try to predict the nature of the solution without solving the problem.
(b) Test your prediction by finding the solution and drawing its graph.
(c) Determine what happens after the sequence of impulses ends.

17. $f(t) = \sum_{k=1}^{20} \delta(t - k\pi)$

18. $f(t) = \sum_{k=1}^{20} (-1)^{k+1}\delta(t - k\pi)$

19. $f(t) = \sum_{k=1}^{20} \delta(t - k\pi/2)$

20. $f(t) = \sum_{k=1}^{20} (-1)^{k+1}\delta(t - k\pi/2)$

21. $f(t) = \sum_{k=1}^{15} \delta[t - (2k-1)\pi]$

22. $f(t) = \sum_{k=1}^{40} (-1)^{k+1}\delta(t - 11k/4)$

23. The position of a certain lightly damped oscillator satisfies the initial value problem

$$y'' + 0.1y' + y = \sum_{k=1}^{20} (-1)^{k+1}\delta(t - k\pi), \qquad y(0) = 0, \quad y'(0) = 0.$$

Observe that, except for the damping term, this problem is the same as Problem 18.
(a) Try to predict the nature of the solution without solving the problem.
(b) Test your prediction by finding the solution and drawing its graph.
(c) Determine what happens after the sequence of impulses ends.

24. Proceed as in Problem 23 for the oscillator satisfying

$$y'' + 0.1y' + y = \sum_{k=1}^{15} \delta[t - (2k-1)\pi], \qquad y(0) = 0, \quad y'(0) = 0.$$

Observe that, except for the damping term, this problem is the same as Problem 21.

25. (a) By the method of variation of parameters show that the solution of the initial value problem

$$y'' + 2y' + 2y = f(t); \qquad y(0) = 0, \quad y'(0) = 0$$

is

$$y = \int_0^t e^{-(t-\tau)} f(\tau) \sin(t - \tau)\, d\tau.$$

(b) Show that if $f(t) = \delta(t - \pi)$, then the solution of part (a) reduces to

$$y = u_\pi(t)e^{-(t-\pi)} \sin(t - \pi).$$

(c) Use a Laplace transform to solve the given initial value problem with $f(t) = \delta(t - \pi)$ and confirm that the solution agrees with the result of part (b).

6.6 The Convolution Integral

Sometimes it is possible to identify a Laplace transform $H(s)$ as the product of two other transforms $F(s)$ and $G(s)$, the latter transforms corresponding to known functions f and g, respectively. In this event, we might anticipate that $H(s)$ would be the transform of the product of f and g. However, this is not the case; in other words, the Laplace transform cannot be commuted with ordinary multiplication. On the other hand, if an appropriately defined "generalized product" is introduced, then the situation changes, as stated in the following theorem.

Theorem 6.6.1 If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ both exist for $s > a \geq 0$, then

$$H(s) = F(s)G(s) = \mathcal{L}\{h(t)\}, \qquad s > a, \tag{1}$$

where

$$h(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)g(t-\tau)\,d\tau. \tag{2}$$

The function h is known as the convolution of f and g; the integrals in Eq. (2) are known as convolution integrals.

The equality of the two integrals in Eq. (2) follows by making the change of variable $t - \tau = \xi$ in the first integral. Before giving the proof of this theorem let us make some observations about the convolution integral. According to this theorem, the transform of the convolution of two functions, rather than the transform of their ordinary product, is given by the product of the separate transforms. It is conventional to emphasize that the convolution integral can be thought of as a "generalized product" by writing

$$h(t) = (f * g)(t). \tag{3}$$

In particular, the notation $(f * g)(t)$ serves to indicate the first integral appearing in Eq. (2).

The convolution $f * g$ has many of the properties of ordinary multiplication. For example, it is relatively simple to show that

$$f * g = g * f \qquad \text{(commutative law)} \tag{4}$$
$$f * (g_1 + g_2) = f * g_1 + f * g_2 \qquad \text{(distributive law)} \tag{5}$$
$$(f * g) * h = f * (g * h) \qquad \text{(associative law)} \tag{6}$$
$$f * 0 = 0 * f = 0. \tag{7}$$

The proofs of these properties are left to the reader. However, there are other properties of ordinary multiplication that the convolution integral does not have. For example, it is not true in general that $f * 1$ is equal to f. To see this, note that

$$(f * 1)(t) = \int_0^t f(t-\tau)\cdot 1\,d\tau = \int_0^t f(t-\tau)\,d\tau.$$

If, for example, $f(t) = \cos t$, then

$$\begin{aligned}(f * 1)(t) &= \int_0^t \cos(t-\tau)\,d\tau = -\sin(t-\tau)\Big|_{\tau=0}^{\tau=t} \\ &= -\sin 0 + \sin t \\ &= \sin t.\end{aligned}$$

Clearly, $(f * 1)(t) \neq f(t)$. Similarly, it may not be true that $f * f$ is nonnegative. See Problem 3 for an example.

Convolution integrals arise in various applications in which the behavior of the system at time t depends not only on its state at time t, but on its past history as well. Systems of this kind are sometimes called hereditary systems and occur in such diverse fields as neutron transport, viscoelasticity, and population dynamics.

Turning now to the proof of Theorem 6.6.1, we note first that if

$$F(s) = \int_0^\infty e^{-s\xi} f(\xi)\, d\xi$$

and

$$G(s) = \int_0^\infty e^{-s\eta} g(\eta)\, d\eta,$$

then

$$F(s)G(s) = \int_0^\infty e^{-s\xi} f(\xi)\, d\xi \int_0^\infty e^{-s\eta} g(\eta)\, d\eta. \tag{8}$$

Since the integrand of the first integral does not depend on the integration variable of the second, we can write $F(s)G(s)$ as an iterated integral,

$$F(s)G(s) = \int_0^\infty g(\eta)\, d\eta \int_0^\infty e^{-s(\xi+\eta)} f(\xi)\, d\xi. \tag{9}$$

This expression can be put into a more convenient form by introducing new variables of integration. First let $\xi = t - \eta$, for fixed η. Then the integral with respect to ξ in Eq. (9) is transformed into one with respect to t; hence

$$F(s)G(s) = \int_0^\infty g(\eta)\, d\eta \int_\eta^\infty e^{-st} f(t-\eta)\, dt. \tag{10}$$

Next let $\eta = \tau$; then Eq. (10) becomes

$$F(s)G(s) = \int_0^\infty g(\tau)\, d\tau \int_\tau^\infty e^{-st} f(t-\tau)\, dt. \tag{11}$$

The integral on the right side of Eq. (11) is carried out over the shaded wedge-shaped region extending to infinity in the $t\tau$-plane shown in Figure 6.6.1. Assuming that the order of integration can be reversed, we finally obtain

$$F(s)G(s) = \int_0^\infty e^{-st}\, dt \int_0^t f(t-\tau) g(\tau)\, d\tau, \tag{12}$$

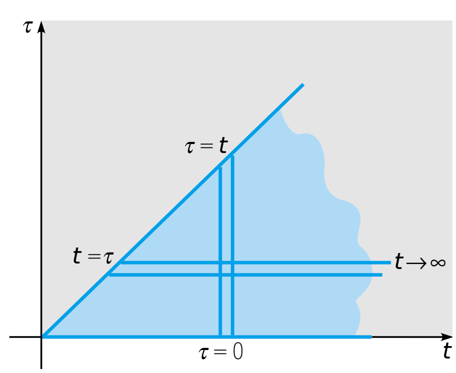

FIGURE 6.6.1 Region of integration in $F(s)G(s)$.

or

$$F(s)G(s) = \int_0^\infty e^{-st}h(t)\,dt$$
$$= \mathcal{L}\{h(t)\}, \tag{13}$$

where $h(t)$ is defined by Eq. (2). This completes the proof of Theorem 6.6.1.

EXAMPLE 1

Find the inverse transform of

$$H(s) = \frac{a}{s^2(s^2+a^2)}. \tag{14}$$

It is convenient to think of $H(s)$ as the product of s^{-2} and $a/(s^2+a^2)$, which, according to lines 3 and 5 of Table 6.2.1, are the transforms of t and $\sin at$, respectively. Hence, by Theorem 6.6.1, the inverse transform of $H(s)$ is

$$h(t) = \int_0^t (t-\tau)\sin a\tau\,d\tau = \frac{at - \sin at}{a^2}. \tag{15}$$

You can verify that the same result is obtained if $h(t)$ is written in the alternate form

$$h(t) = \int_0^t \tau \sin a(t-\tau)\,d\tau,$$

which confirms Eq. (2) in this case. Of course, $h(t)$ can also be found by expanding $H(s)$ in partial fractions.

EXAMPLE 2

Find the solution of the initial value problem

$$y'' + 4y = g(t), \tag{16}$$

$$y(0) = 3, \qquad y'(0) = -1. \tag{17}$$

By taking the Laplace transform of the differential equation and using the initial conditions, we obtain

$$s^2Y(s) - 3s + 1 + 4Y(s) = G(s),$$

or

$$Y(s) = \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4}. \tag{18}$$

Observe that the first and second terms on the right side of Eq. (18) contain the dependence of $Y(s)$ on the initial conditions and forcing function, respectively. It is convenient to write $Y(s)$ in the form

$$Y(s) = 3\frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}G(s). \tag{19}$$

Then, using lines 5 and 6 of Table 6.2.1 and Theorem 6.6.1, we obtain

$$y = 3\cos 2t - \tfrac{1}{2}\sin 2t + \tfrac{1}{2}\int_0^t \sin 2(t-\tau)g(\tau)\,d\tau. \tag{20}$$

If a specific forcing function g is given, then the integral in Eq. (20) can be evaluated (by numerical means, if necessary).

Example 2 illustrates the power of the convolution integral as a tool for writing the solution of an initial value problem in terms of an integral. In fact, it is possible to proceed in much the same way in more general problems. Consider the problem consisting of the differential equation

$$ay'' + by' + cy = g(t), \tag{21}$$

where a, b, and c are real constants and g is a given function, together with the initial conditions

$$y(0) = y_0, \qquad y'(0) = y_0'. \tag{22}$$

The transform approach yields some important insights concerning the structure of the solution of any problem of this type.

The initial value problem (21), (22) is often referred to as an input–output problem. The coefficients a, b, and c describe the properties of some physical system, and $g(t)$ is the input to the system. The values y_0 and y_0' describe the initial state, and the solution y is the output at time t.

By taking the Laplace transform of Eq. (21) and using the initial conditions (22), we obtain

$$(as^2 + bs + c)Y(s) - (as + b)y_0 - ay_0' = G(s).$$

If we let

$$\Phi(s) = \frac{(as + b)y_0 + ay_0'}{as^2 + bs + c}, \qquad \Psi(s) = \frac{G(s)}{as^2 + bs + c}, \tag{23}$$

then we can write

$$Y(s) = \Phi(s) + \Psi(s). \tag{24}$$

Consequently,

$$y = \phi(t) + \psi(t), \tag{25}$$

where $\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\}$ and $\psi(t) = \mathcal{L}^{-1}\{\Psi(s)\}$. Observe that $y = \phi(t)$ is a solution of the initial value problem

$$ay'' + by' + cy = 0, \qquad y(0) = y_0, \quad y'(0) = y_0', \tag{26}$$

obtained from Eqs. (21) and (22) by setting $g(t)$ equal to zero. Similarly, $y = \psi(t)$ is the solution of

$$ay'' + by' + cy = g(t), \qquad y(0) = 0, \quad y'(0) = 0, \tag{27}$$

in which the initial values y_0 and y_0' are each replaced by zero.

Once specific values of a, b, and c are given, we can find $\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\}$ by using Table 6.2.1, possibly in conjunction with a translation or a partial fraction expansion. To find $\psi(t) = \mathcal{L}^{-1}\{\Psi(s)\}$ it is convenient to write $\Psi(s)$ as

$$\Psi(s) = H(s)G(s), \tag{28}$$

where $H(s) = (as^2 + bs + c)^{-1}$. The function H is known as the **transfer function**[3] and depends only on the properties of the system under consideration; that is, $H(s)$ is determined entirely by the coefficients a, b, and c. On the other hand, $G(s)$ depends only on the external excitation $g(t)$ that is applied to the system. By the convolution theorem we can write

$$\psi(t) = \mathcal{L}^{-1}\{H(s)G(s)\} = \int_0^t h(t-\tau)g(\tau)\,d\tau, \tag{29}$$

where $h(t) = \mathcal{L}^{-1}\{H(s)\}$, and $g(t)$ is the given forcing function.

To obtain a better understanding of the significance of $h(t)$, we consider the case in which $G(s) = 1$; consequently, $g(t) = \delta(t)$ and $\Psi(s) = H(s)$. This means that $y = h(t)$ is the solution of the initial value problem

$$ay'' + by' + cy = \delta(t), \qquad y(0) = 0, \quad y'(0) = 0, \tag{30}$$

obtained from Eq. (27) by replacing $g(t)$ by $\delta(t)$. Thus $h(t)$ is the response of the system to a unit impulse applied at $t = 0$, and it is natural to call $h(t)$ the **impulse response** of the system. Equation (29) then says that $\psi(t)$ is the convolution of the impulse response and the forcing function.

Referring to Example 2, we note that in that case the transfer function is $H(s) = 1/(s^2 + 4)$, and the impulse response is $h(t) = (\sin 2t)/2$. Also, the first two terms on the right side of Eq. (20) constitute the function $\phi(t)$, the solution of the corresponding homogeneous equation that satisfies the given initial conditions.

PROBLEMS

1. Establish the commutative, distributive, and associative properties of the convolution integral.
   (a) $f * g = g * f$
   (b) $f * (g_1 + g_2) = f * g_1 + f * g_2$
   (c) $f * (g * h) = (f * g) * h$
2. Find an example different from the one in the text showing that $(f * 1)(t)$ need not be equal to $f(t)$.
3. Show, by means of the example $f(t) = \sin t$, that $f * f$ is not necessarily nonnegative.

In each of Problems 4 through 7 find the Laplace transform of the given function.

4. $f(t) = \int_0^t (t-\tau)^2 \cos 2\tau \, d\tau$
5. $f(t) = \int_0^t e^{-(t-\tau)} \sin \tau \, d\tau$
6. $f(t) = \int_0^t (t-\tau)e^{\tau} \, d\tau$
7. $f(t) = \int_0^t \sin(t-\tau) \cos \tau \, d\tau$

In each of Problems 8 through 11 find the inverse Laplace transform of the given function by using the convolution theorem.

8. $F(s) = \dfrac{1}{s^4(s^2+1)}$
9. $F(s) = \dfrac{s}{(s+1)(s^2+4)}$
10. $F(s) = \dfrac{1}{(s+1)^2(s^2+4)}$
11. $F(s) = \dfrac{G(s)}{s^2+1}$

[3] This terminology arises from the fact that $H(s)$ is the ratio of the transforms of the output and the input of the problem (27).

In each of Problems 12 through 19 express the solution of the given initial value problem in terms of a convolution integral.

12. $y'' + \omega^2 y = g(t); \quad y(0) = 0, \quad y'(0) = 1$
13. $y'' + 2y' + 2y = \sin \alpha t; \quad y(0) = 0, \quad y'(0) = 0$
14. $4y'' + 4y' + 17y = g(t); \quad y(0) = 0, \quad y'(0) = 0$
15. $y'' + y' + \frac{5}{4}y = 1 - u_\pi(t); \quad y(0) = 1, \quad y'(0) = -1$
16. $y'' + 4y' + 4y = g(t); \quad y(0) = 2, \quad y'(0) = -3$
17. $y'' + 3y' + 2y = \cos \alpha t; \quad y(0) = 1, \quad y'(0) = 0$
18. $y^{\text{iv}} - y = g(t); \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$
19. $y^{\text{iv}} + 5y'' + 4y = g(t); \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$
20. Consider the equation

$$\phi(t) + \int_0^t k(t - \xi)\phi(\xi)\, d\xi = f(t),$$

in which f and k are known functions, and ϕ is to be determined. Since the unknown function ϕ appears under an integral sign, the given equation is called an **integral equation**; in particular, it belongs to a class of integral equations known as Volterra integral equations. Take the Laplace transform of the given integral equation and obtain an expression for $\mathcal{L}\{\phi(t)\}$ in terms of the transforms $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{k(t)\}$ of the given functions f and k. The inverse transform of $\mathcal{L}\{\phi(t)\}$ is the solution of the original integral equation.

21. Consider the Volterra integral equation (see Problem 20)

$$\phi(t) + \int_0^t (t - \xi)\phi(\xi)\, d\xi = \sin 2t.$$

(a) Show that if u is a function such that $u''(t) = \phi(t)$, then

$$u''(t) + u(t) - tu'(0) - u(0) = \sin 2t.$$

(b) Show that the given integral equation is equivalent to the initial value problem

$$u''(t) + u(t) = \sin 2t; \qquad u(0) = 0, \quad u'(0) = 0.$$

(c) Solve the given integral equation by using the Laplace transform.
(d) Solve the initial value problem of part (b) and verify that the solution is the same as that obtained in part (c).

22. **The Tautochrone.** A problem of interest in the history of mathematics is that of finding the *tautochrone*—the curve down which a particle will slide freely under gravity alone, reaching the bottom in the same time regardless of its starting point on the curve. This problem arose in the construction of a clock pendulum whose period is independent of the amplitude of its motion. The tautochrone was found by Christian Huygens (1629–1695) in 1673 by geometrical methods, and later by Leibniz and Jakob Bernoulli using analytical arguments. Bernoulli's solution (in 1690) was one of the first occasions on which a differential equation was explicitly solved.

The geometrical configuration is shown in Figure 6.6.2. The starting point $P(a, b)$ is joined to the terminal point $(0, 0)$ by the arc C. Arc length s is measured from the origin, and $f(y)$ denotes the rate of change of s with respect to y:

$$f(y) = \frac{ds}{dy} = \left[1 + \left(\frac{dx}{dy}\right)^2\right]^{1/2}. \tag{i}$$

Then it follows from the principle of conservation of energy that the time $T(b)$ required for a particle to slide from P to the origin is

$$T(b) = \frac{1}{\sqrt{2g}} \int_0^b \frac{f(y)}{\sqrt{b - y}}\, dy. \tag{ii}$$

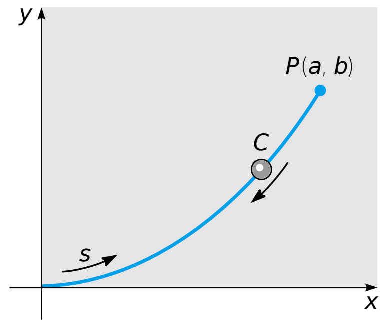

FIGURE 6.6.2 The tautochrone.

(a) Assume that $T(b) = T_0$, a constant, for each b. By taking the Laplace transform of Eq. (ii) in this case and using the convolution theorem, show that

$$F(s) = \sqrt{\frac{2g}{\pi}} \frac{T_0}{\sqrt{s}}; \tag{iii}$$

then show that

$$f(y) = \sqrt{\frac{2g}{\pi}} \frac{T_0}{\sqrt{y}}. \tag{iv}$$

Hint: See Problem 27 of Section 6.1.
(b) Combining Eqs. (i) and (iv), show that

$$\frac{dx}{dy} = \sqrt{\frac{2\alpha - y}{y}}, \tag{v}$$

where $\alpha = gT_0^2/\pi^2$.
(c) Use the substitution $y = 2\alpha \sin^2(\theta/2)$ to solve Eq. (v), and show that

$$x = \alpha(\theta + \sin\theta), \qquad y = \alpha(1 - \cos\theta). \tag{vi}$$

Equations (vi) can be identified as parametric equations of a cycloid. Thus the tautochrone is an arc of a cycloid.

REFERENCES

The books listed below contain additional information on the Laplace transform and its applications:

Churchill, R. V., *Operational Mathematics* (3rd ed.) (New York: McGraw-Hill, 1971).

Doetsch, G., Nader, W. (tr.), *Introduction to the Theory and Application of the Laplace Transform* (New York: Springer-Verlag, 1974).

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Kuhfittig, P. K. F., *Introduction to the Laplace Transform* (New York: Plenum, 1978).

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Rainville, E. D., *The Laplace Transform: An Introduction* (New York: Macmillan, 1963).

Each of the books just mentioned contains a table of transforms. Extensive tables are also available; see, for example:

Erdelyi, A. (ed.), *Tables of Integral Transforms* (Vol. 1) (New York: McGraw-Hill, 1954).

Roberts, G. E., and Kaufman, H., *Table of Laplace Transforms* (Philadelphia: Saunders, 1966).

A further discussion of generalized functions can be found in:

Lighthill, M. J., *Fourier Analysis and Generalized Functions* (London: Cambridge University Press, 1958).